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Collision-Free and Continuous-Curvature Path Planning for Car-Like Robots

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Abstract: *This paper presents a set of paths, called* bi-elementary paths. *These paths are smooth and feasible for a car-like robot (i.e. their tangent direction is continuous and they respect a minimum turning radius constraint), and they can be followed by a real vehicle without stopping (i.e. they have a continuous curvature profile) — which is not the case of Dubins' curves. These paths are composed of arcs of clothoid (a clothoid is a curve whose curvature is a linear function of its arc length), and are used to define a simplified, i.e. non complete, planner. This simplified planner is, in turn, used in two global planning schemes, namely the Ariadne's Clew algorithm and the Probabilistic Path Planning. This paper proves an important property of the bi-elementary paths, from which the completeness of the two global planners is deduced.*

Keywords — mobile-robot, path-planning, non-holonomic-system, continuous-curvature paths, clothoids, global planner, completeness.

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1 Introduction

To the best of our knowledge, current path planners for car-like robots (i.e. planners taking into account the non-collision and non-holonomic kinematic constraints of the robot) return paths made of line segments and circular arcs [2, 6, 9, 15]. These paths do not have a continuous curvature profile and thus, if a real vehicle were to follow precisely one of them, it would have to stop at each curvature discontinuity to reorient its front wheel(s). Continuous curvature paths have been previously considered, but usually to generate paths for car-like robots (path generation deals with the kinematic constraints only, not the obstacles) [11, 7, 3]. Fleury at al. did present a continuous curvature path planner in [4], but it involved a mobile robot of the Hilare family, which was not a car-like robot (it did not have a minimum turning radius constraint).

A first continuous-curvature path planner (CCPP) was presented in a previous paper [12]. This paper presented the concept of *elementary paths*, which are a pair of symmetric clothoid arcs (a clothoid is a curve whose curvature is a linear function of its arc length). It also presented a simplified (or *local*) planner based on *bi-elementary paths*, i.e. couples of elementary paths. As this local planner is not complete, a *global* (i.e. complete) planner called CCPP was build using this local planner and a global planning method: the Ariadne's Clew algorithm [10].

The aim of this paper is to prove the completeness of CCPP and to show that the local planner can be used with other global planning methods in order to obtain a complete planner. An example is given using the Probabilistic Path Planning method PPP [15]. In order to demonstrate the completeness (with respect to the set of paths considered) of both CCPP and PPP, we prove a continuity property on the existence of a bi-elementary path between two configurations.

Outline of the paper. After a short presentation of the planning problem (§2), the elementary paths and bi-elementary paths are defined (§3 and the different properties required to show the completeness of CCPP and PPP are proved. Then, the simplified planner returning a bi-elementary path between two configurations is described (§4). CCPP, i.e. the path planner based on the Ariadne's Clew algorithm, is presented with the proof of its completeness (§5). Finally, the same work is done for the planner using the Probabilistic Path Planner method (§6).

2 Statement of the Problem

Our goal is to design a path planner for a fast car-like robot. This induces two main problems: the paths generated by this planner have to avoid obstacles of the environment, but the vehicle must also be able to follow them precisely (this is not the case for the paths usually generated by path planners nowadays).

We will first recall the constraints to be verified by our car-like robot. Then, the kind of path which can be followed by the robot will be explained and the resulting path planning problem will be stated.

2.1 Feasible and Smooth Paths

We model a car-like robot moving on a plane surface. Thus, the workspace $\mathcal{W}$ of our robot $\mathcal{A}$ is equivalent to a sub-space of $\mathbb{R}^2$. In this workspace, the obstacles are represented by a set of polygonal regions noted $\mathcal{B}_j$, $j \in \{1, ..., n_B\}$. A configuration of $\mathcal{A}$ is defined by the coordinates (x, y) of the rear axle midpoint R and the orientation θ of $\mathcal{A}$, i.e. the angle between the x-axis and the main axis of $\mathcal{A}$. The configuration space (the $xy\theta$-space) is denoted by $\mathcal{C}$, $\mathcal{C}_{\text{collision}}$ is the subset of the configurations for which $\mathcal{A}$ collides one of the obstacle's $\mathcal{B}_j$, $j \in \{1, ..., n_B\}$, $\mathcal{C}_{\text{free}}$ is the complement of $\mathcal{C}_{\text{collision}}$ in $\mathcal{C}$, and $\mathcal{C}^{\varepsilon}_{\text{free}}$ is the subset of the configurations of $\mathcal{C}_{\text{free}}$ whose distance (in $\mathcal{C}$) to $\mathcal{C}_{\text{collision}}$ is greater than ε.

The movement of $\mathcal{A}$ is constrained physically for two reasons: the non-sliding of the wheels and the bounds on the front wheels' orientation. The non-sliding of the rear

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wheels implies that the main axis of $\mathcal{A}$ remains collinear to the derivate vector of the position of R. This constraint (called *orientation constraint*) can be written:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \quad (1)$$

The non-sliding of the front wheels and the bounds on their orientation implies a minimum on the turning radius of $\mathcal{A}$, i.e. bounds on the curvature of the curve followed by R. Thus, the following *turning radius constraint* holds:

$$\dot{x}^2 + \dot{y}^2 - {\rho_{\min}}^2\dot{\theta}^2 \geq 0 \quad (2)$$

A path Π for $\mathcal{A}$ is a continuous curve in $\mathcal{C}$ but, when it respects the constraint (1), it is equivalent to its projection in $\mathcal{W}$, which is called its xy-curve [8]. This path is *feasible* if and only if a) the tangent direction of its xy-curve is piecewise continuous with opposite semi-tangents at the cusp points (which are the change of the direction of motion), and b) the curvature of its xy-curve is at each point less than $1/\rho_{\min}$. As our robot will be fast, we prefer to have a *smooth* path (i.e. without manœuvre), so its tangent direction has to be continuous. A smooth path can be represented by the curvature function of its xy-curve, what we call its *curvature profile*.

Usual path planners return paths made of straight segments connected with tangential circular arcs of minimum radius. If these paths are feasible, they cannot be followed by a real vehicle without stopping at each curvature discontinuity to reorient the front wheels. This is not acceptable for a fast vehicle. For this reason, the generated path has to be a *continuous-curvature* one, i.e. its curvature profile has to be continuous.

2.2 Our Path-Planning Problem

In summary, given a starting configuration q_s and a goal configuration q_g (both in $\mathcal{C}_{\text{free}}$), we need to plan a path Π such that:

- Π verifies the end conditions, i.e. it is a curve of $\mathcal{C}$ linking q_s to q_g;
- Π is a feasible, smooth and continuous-curvature path;
- Π is collision-free, i.e. it is a curve included in $\mathcal{C}_{\text{free}}$.

Such a path Π is said to be a *solution path*. As the set of the solution paths is too wide to be considered, we limit the search of our planner to a subset of it: we only search paths made of a sequence of *elementary paths*.

3 Elementary and Bi-Elementary Paths

In this section, the definition of the elementary paths is given, which leads to the definition of the bi-elementary paths. Then, the definitions and properties of these paths, which are needed to prove the completeness of our global planners, will be presented. Due to lack of place, the proof of these properties can be hard to understand. The complete proofs are detailed in a research report [14].

3.1 Definitions of the Elementary and Bi-Elementary Paths

As we saw in 2.1, a feasible path Π in $\mathcal{C}$ is equivalent to its projection in $\mathcal{W}$. If Π is smooth, it can be represented by its length and its curvature profile (see §3 in [7], or §3.2 in [12]). An elementary path, which is made of two symmetric arcs of clothoid (see Fig. 1), can therefore be defined as follows:

Definition 1 (Elementary Path) *A smooth path of length l is an* elementary path *if and only if its curvature profile κ verifies:*

$$\exists\sigma \in I\!R \;/\; \forall s \in [0, l/2],\; \kappa(s) = \kappa(l-s) = \sigma s$$

Thus, an elementary path can be represented by a pair $(l, \sigma) \in I\!R^2$, where σ is called the *sharpness* of the path.

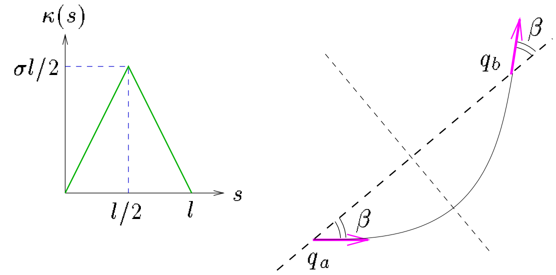

Figure 1: $\kappa(s)$ and xy-curve of an elementary path.

An elementary path is a symmetric path (its curvature profile is a symmetric function), and it can therefore only link two configurations q_a and q_b that are *symmetric*. This means that the orientations of q_a and q_b are symmetric with respect to the line joining their position (see Fig. 1, the proof is given in [7], proposition 1). Formally, the coordinates of q_a and q_b must verify:

$$(x_b - x_a)\sin\frac{\theta_b + \theta_a}{2} = (y_b - y_a)\cos\frac{\theta_b + \theta_a}{2}$$

The *deflection* of an elementary path is the orientation variation along this path, here $\theta_b - \theta_a = 2\beta$ [7]. Such a path is feasible if and only if it respects the turning radius constraint (2). As its curvature is maximum (or minimum) at its middle configuration, it is feasible if and only if $\sigma l/2 \leq 1/\rho_{\min}$.

We define bi-elementary paths as sequences of two elementary paths (cf. Figure 2):

Definition 2 (Bi-Elementary Path) *A smooth path of length l is a* bi-elementary path *if and only there exists three reals σ_0, σ_1, l_i such that:*

$$\begin{cases} \forall s \in [0, l_i/2], & \kappa(s) = \kappa(l_i - s) = \sigma_0 s \\ \forall s \in [(l+l_i)/2, l], & \kappa(s) = \kappa(l + l_i - s) = -\sigma_1(l - s) \end{cases}$$

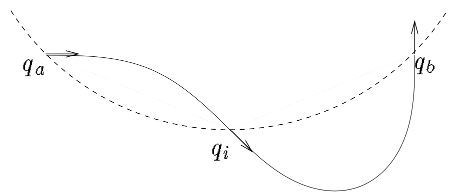

Figure 2: a bi-elementary path.

As we will see in the following, these paths can link any two configurations and are therefore much more interesting for planning. To obtain the property of these bi-elementary paths which allows to prove the completeness of our planners, we have to state first a few properties of the elementary paths.

3.2 Properties of Elementary Paths

We will first give a characterization of the set of configurations which can be linked using a feasible elementary path (i.e. an elementary path which respects the turning radius constraint (2)). Therefore, we have to find the conditions two configurations have to respect for a feasible elementary path linking them to exist.

Theorem 1 (Existence of an Elementary Path) *Let $q_a = (x_a, y_a, \theta_a)$ and $q_b = (x_b, y_b, \theta_b)$ be two symmetric configurations. Let us rewrite the vector $(x_b - x_a, y_b - y_a)$ in polar coordinates as $(r\cos(\theta_a + \beta), r\sin(\theta_a + \beta))$, with $r \in [0, +\infty[$ and $\beta \in]-\pi, \pi]$. A feasible elementary path linking q_a to q_b exists if and only if r and β are such that:*

$$\text{or}\begin{cases} |\beta| \in [0, \Theta_{root}[& \text{and} \quad r \geq 4\rho_{min}\sqrt{|\beta|}D_1(|\beta|) \\ |\beta| = \Theta_{root} & \text{and} \quad r = 0 \end{cases}$$

where D_1 is the function defined over $[0, \pi]$ as:

$$D_1(\alpha) = \cos\alpha \int_0^{\sqrt{\alpha}} \cos u^2 du + \sin\alpha \int_0^{\sqrt{\alpha}} \sin u^2 du$$

and where Θ_{root} is the unique root of D_1 over $]0, \pi]$ (cf. Fig. 3).

We then note $q_a \mathcal{R} q_b$ (q_a reach q_b).

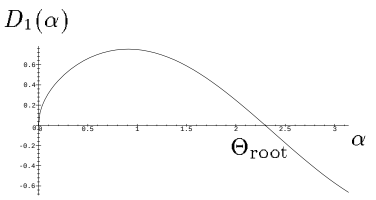

Figure 3: the function D_1 along $[0, \pi]$.

Proof: By rotation and translation, the configuration q_a can be transformed into the configuration $(0, 0, 0)$. Therefore, q_a will be supposed to be equal to $(0, 0, 0)$ in the following, and 2β will be the path's deflection.

To prove this theorem, we will suppose that there exists a feasible elementary path Ψ linking $q_a = (0, 0, 0)$ to q_b, find the constraint q_b has to verify and then calculate the parameters of Ψ depending on q_b. Thus, we will have proved that the existence of Ψ implies the constraint and that, if this constraint is verified, Ψ exists. To obtain the constraint, we have to consider three different cases.

1. First, let us suppose that r and β are not null (this means that the position of q_b is not on the half-line $[Ox)$). If l and σ are the length and the sharpness of the elementary path Ψ linking $(0, 0, 0)$ ($= q_a$) to q_b, the coordinates of the middle configuration of Ψ are (let ξ be the sign of σ):

$$q_m : \begin{cases} x &= \int_0^{l/2} \cos(\sigma t^2/2)dt &= \sqrt{2/|\sigma|} \int_0^{\sqrt{|\beta|}} \cos u^2 du \\ y &= \int_0^{l/2} \sin(\sigma t^2/2)dt &= \sqrt{2/|\sigma|}\, \xi \int_0^{\sqrt{|\beta|}} \sin u^2 du \\ \theta &= \sigma(l/2)^2/2 &= \sigma l^2/8 \end{cases}$$

The proposition 1 of the paper [7] prove that the position P_m of this configuration is located on the bisecting line of the segment $[P_a, P_b]$, were P_a is the position of q_a and P_b the position of q_b. Therefore, we have $\overrightarrow{P_aP_m}.\overrightarrow{P_aP_b} = r^2/2$, which can be written as $r\sqrt{2/|\sigma|}D_1(|\beta|) = r^2/2$ (using the coordinates of q_m we just obtained). This implies that $D_1(|\beta|)$ is strictly positive (as r is not null), i.e. that β is in the set $]-\Theta_{\text{root}}, \Theta_{\text{root}}[$ (cf. Fig. 3), and that $|\sigma| = 8\, D_1(|\beta|)^2/r^2$.

On another hand, the same proposition 1 prove that the orientation of q_m is perpendicular to the bisecting line of $[P_a, P_b]$. Therefore, $\theta = \sigma l^2/8 = \beta$ and the parameters of Ψ can be deduced from r and β using the equations:

$$\begin{cases} \sigma &= 8\,\text{sgn}(\beta)\, D_1(|\beta|)^2/r^2 \\ l &= 2\sqrt{2\beta/\sigma} \end{cases} \tag{3}$$

To conclude, Ψ is feasible if and only if $|\sigma|.l/2 \leq 1/\rho_{\text{min}}$, which can be rewritten as $r \geq 4\,\rho_{\text{min}}\,\sqrt{|\beta|}\, D_1(|\beta|)$, for $\beta \in]-\Theta_{\text{root}}, \Theta_{\text{root}}[\setminus\{0\}$.

2. We now suppose r null and β not null (q_a and q_b have the same position but are not equal). The constraint $\sigma l^2/8 = \beta$ still holds, but the other becomes $\sqrt{2/|\sigma|}D_1(|\beta|) = 0$ (see [14] for more details). Thus, $D_1(|\beta|)$ has to be null, and $|\beta| = \Theta_{\text{root}}$ (β is supposed not null). An infinity of feasible elementary paths linking q_a to q_b can then be found.

3. At last, if β is null, the position of q_b is on the half-line [Ox) and its orientation is the same as q_a's. In this case, the elementary path is a segment, and q_b can always be reached ($q_a = (0, 0, 0)$): r can have any value of $[0, +\infty[$.

The cases 1 and 3 give the first line of the theorem's constraint, the case 2 gives the second line. ■

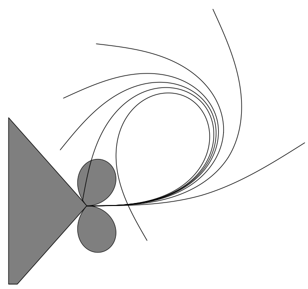

Figure 4: examples of elementary paths.

From this theorem, we can deduce the set $\mathcal{Q}_{\text{reach}}(q_a)$ of configurations reachable from a configuration q_a with a feasible elementary path: $\mathcal{Q}_{\text{reach}}(q_a) = \{q \in \mathcal{C}_{\text{free}}\ /\ q_a\ \mathcal{R}\ q\}$. Figure 4 gives, starting from configuration $(0, 0, 0)$, the set of the reachable symmetric configurations (white zone), and some examples of elementary paths. This helps to understand why the half-deflection β has to be bounded by Θ_{root}: with a deflection of $2\Theta_{\text{root}}$, the elementary path comes back to its starting position; if the deflection is greater than $2\Theta_{\text{root}}$, the elementary path makes a loop and comes back to the half-plane $(x > 0)$.

The set $\mathcal{Q}_{\text{reach}}(q_a)$ is not open-ended. We want to prove that the existence of a feasible and non-colliding elementary path linking two configurations can be extended to a neighbourhood of both configurations, assuming certain conditions. Therefore, as neighbourhoods are concerned, the set of configurations considered has to be open-ended. Thus, let us set the following definition:

Definition 3 *Let Ψ be an elementary path, and $q_a = (x_a, y_a, \theta_a)$ and $q_b = (x_b, y_b, \theta_b)$ be its starting and final configurations. We also rewrite the vector $(x_b - x_a, y_b - y_a)$ in polar coordinates as $(r\cos(\theta_a + \beta), r\sin(\theta_a + \beta))$, with $r \in [0, +\infty[$*

and $\beta \in]{-\pi}, \pi]$. Ψ is said to be extreme *if and only if it is feasible but $|\beta| = \Theta_{root}$ or $r = 4\,\rho_{min}\,\sqrt{|\beta|}\,D_1(|\beta|)$.*

Therefore, if Ψ is a non-extreme (feasible) elementary path linking q_a to q_b, any configuration in a neighbourhood (small enough) of q_b can be reached from q_a with a feasible elementary path. This is true because this neighbourhood can be totally included in $\mathcal{Q}_{\text{reach}}(q_a)$, if it is small enough.

On the other hand, we must take obstacles into account.

Definition 4 *Given a configuration q_a and a subset $\mathcal{S}$ of $\mathcal{C}$, we call $\mathcal{RC}_1(q_a, \mathcal{S})$ the set of configurations which can be reached from q_a with a feasible elementary path included in $\mathcal{S}$.*

Thus, each time we will write in the following $q_b \in \mathcal{RC}_1(q_a, \mathcal{S})$, this will mean *there exists a feasible elementary path included in $\mathcal{S}$, linking q_a to q_b.* We have found the equations giving $\mathcal{Q}_{\text{reach}}(q_a) = \mathcal{RC}_1(q_a, \mathcal{C})$, for all $q_a \in \mathcal{C}$. We now prove a second main property of elementary paths:

Theorem 2 *The workspace $\mathcal{W}$ of $\mathcal{A}$ is assumed to be bounded, and $\varepsilon > 0$ is a real constant. Let q_a and q_b be two configurations linked by a non-extreme (feasible) elementary path Ψ in $\mathcal{C}^{\varepsilon}_{free}$. There exists $\delta_1 > 0$ such that:*

$$\forall q_a', q_b' \in \mathcal{C}_{free},$$
$$(d(q_a', q_a) < \delta_1) \wedge (d(q_b', q_b) < \delta_1) \wedge$$
$$(q_a' \text{ and } q_b' \text{ symmetric}) \Rightarrow q_b' \in \mathcal{RC}_1(q_a', \mathcal{C}_{free})$$

Proof: Assuming that the hypotheses of this theorem are true, we search the constraints $\delta_1 > 0$ has to verify for the implication to be true.

Let q_a' and q_b' be two symmetric configurations of $\mathcal{C}_{\text{free}}$ such that $\mathrm{d}(q_a', q_a) < \delta_1$ and $\mathrm{d}(q_b', q_b) < \delta_1$. Let r be the length and $\theta_a + \beta$ the orientation of the vector $\overrightarrow{P_a P_b}$, joining the respective positions P_a and P_b of the configurations q_a and q_b (same notations as in the Theorem 1). Similarly, r' and β' are defined concerning q_a' and q_b'.

It is easily shown that $|r' - r| < 2\delta_1$ and $|\beta' - \beta| < \delta_1$. As Ψ is a non-extreme elementary path, we have $|\beta| < \Theta_{\text{root}}$ and $r > 4\,\rho_{\min}\,\sqrt{|\beta|}\,D_1(|\beta|)$. As the function $\beta \longmapsto \sqrt{|\beta|}D_1(|\beta|)$ is continuously derivable along $]{-\pi}, \pi]$, with a derivate bounded by $3/2$, to have the similar inequalities concerning β' and r', δ_1 must verify the following condition:

$$0 < \delta_1 < \min\left(\frac{\Theta_{\text{root}} - |\beta|}{2}, \frac{r - 16\rho_{\min}\sqrt{|\beta|}D_1(|\beta|)}{2 + 24\rho_{\min}}\right) \quad (4)$$

Then, there exists a non-extreme elementary path Ψ' linking q_a' to q_b' in $\mathcal{C}$. This path is included in $\mathcal{C}_{\text{free}}$ if we can show that it stays at a distance less than ε from the path Ψ. If the length and sharpness of Ψ are l and σ, and those of Ψ' are l' and σ', we note γ the quotient of the lengths: $\gamma = l'/l$. The distance between the two paths is less than the maximum, for $s \in [0, l]$, of the distances between the configuration $q(s)$ of Ψ at length s and the configuration $q'(s')$ of Ψ' at length $s' = \gamma s$. As the paths are symmetric, and the distance inequalities of their starting and final configurations are also symmetric, the calculus of the previous maximum distance can be limited to the first part of the path (i.e. we can take $s \in [0, l/2]$). Therefore, we have to find the maximum of the distance between $q'(s') = (x'(s'), y'(s'), \theta'(s'))$ and $q(s) = (x(s), y(s), \theta(s))$, for $s \in [0, l/2]$ and $s' = \gamma s$. First, we note that:

$$\begin{aligned} \theta'(s') - \theta(s) &= \theta_a' + \sigma' s'^2/2 - \theta_a - \sigma s^2/2 \\ &= \theta_a' - \theta_a + (\sigma'\gamma^2 - \sigma)s^2/2 \end{aligned}$$

and, using the second line of the equation (3):

$$\sigma'\gamma^2 - \sigma = (\sigma' l'^2 - \sigma l^2)/l^2 = 8(\beta' - \beta)/l^2$$

Thus, as $|\beta' - \beta| < \delta_1$, $|\sigma'\gamma^2 - \sigma| < 8\delta_1/l^2$, and $|\theta'(s') - \theta(s)| < |\theta_a' - \theta_a| + 4\delta_1 s^2/l^2 \leq 2\delta_1$ $(s \leq l/2)$. Moreover:

$$\begin{aligned} &|x'(s') - x(s)| \\ &= \left| x_a' - x_a + (\gamma - 1)\int_0^s \cos\theta'(\gamma u)du + \right. \\ &\qquad \left. \int_0^s \left(\cos\theta'(\gamma u) - \cos\theta(u)\right) du \right| \\ &< |x_a' - x_a| + |\gamma - 1|s + \int_0^s |\theta'(\gamma u) - \theta(u)| du \\ &< \delta_1 + |\gamma - 1| l/2 + 2\delta_1 s \\ &< |\gamma - 1| l/2 + (1 + l)\delta_1 \end{aligned}$$

A similar result can be obtained with $|y'(s') - y(s)|$. Thus, we have the following inequalities:

$$\left\{ \begin{array}{lcl} |x'(s') - x(s)| & < & |l' - l|/2 + (1 + l)\delta_1 \\ |y'(s') - y(s)| & < & |l' - l|/2 + (1 + l)\delta_1 \\ |\theta'(s') - \theta(s)| & < & 2\delta_1 \end{array} \right.$$

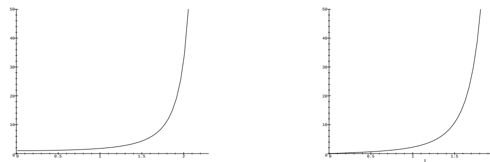

Figure 5: the function f and f' along $[0, \Theta_{\text{root}}[$.

We now need to find a bound on $|l' - l|$. We define a function f along $] - \Theta_{\text{root}}, \Theta_{\text{root}}[$ as $f(0) = 1$ and $f(\alpha) = |\alpha|/D_1(\alpha)^2$ for $\alpha \neq 0$. We have $l^2 = f(\beta)r^2$ and $l'^2 = f(\beta')r'^2$. We can prove (using Taylor polynomials) that f is continuous and derivable in 0, and therefore on $] - \Theta_{\text{root}}, \Theta_{\text{root}}[$. Moreover, f and its derivate f' increase along $[0, \Theta_{\text{root}}[$ (cf. Fig. 5). For this reason, $f(\beta)$ and $f(\beta')$ can be bounded by $M_f = f((|\beta| + \Theta_{\text{root}})/2)$, and $f'(\beta)$ and $f'(\beta')$ by $M_{f'} = f'((|\beta| + \Theta_{\text{root}})/2)$. As l' and l are positive, we can write $|l' - l|^2 \leq |l'^2 - l^2| \leq |f(\beta')r'^2 - f(\beta)r^2| \leq |f(\beta') - f(\beta)|r'^2 + f(\beta)|r'^2 - r^2| < r'^2 M_{f'}\delta_1 + 2(r' + r)M_f\delta_1$.

Finaly, we must note that $\mathcal{W}$ is bounded. Let M be a bound on its bigger dimension, we then have $|l' - l|^2 < (2M^2 M_{f'} + 6MM_f)\delta_1$. This leads us to write $\mathrm{d}(q'(s'), q(s))^2 < (l' - l)^2/2 + (4 + 2(l + 1)^2)\delta_1{}^2 < K\delta_1$, where K is a bound of $M^2 M_{f'} + 3MM_f + \delta_1(4 + 2(2M + 1)^2)$ (for example, using the relation (4), $M^2 M_{f'} + 3MM_f + (2 + (2M + 1)^2)(\Theta_{\text{root}} - |\beta|)$).

To conclude, the implication of the Theorem 2 is true if δ_1 verifies:

$$\delta_1 < \min\left(\frac{\Theta_{\text{root}} - |\beta|}{2}, \frac{r - 16\rho_{\min}\sqrt{|\beta|}D_1(|\beta|)}{2 + 24\rho_{\min}}, \frac{\varepsilon^2}{K}\right)$$

■

3.3 Properties of Bi-Elementary Paths

To prove a theorem similar to Theorem 2, we have to extend the definitions of extreme path and of $\mathcal{RC}_1(q_a, \mathcal{S})$ to the bi-elementary paths.

Definition 5 *A bi-elementary path is said to be* extreme *if one of its two elementary paths is extreme or if its starting and final configurations have the same position.*

Definition 6 *Given a configuration q_a and a subset $\mathcal{S}$ of $\mathcal{C}$, we call $\mathcal{RC}_2(q_a, \mathcal{S})$ the set of configuration which can be reached from q_a with a bi-elementary path included in $\mathcal{S}$.*

We also need to prove a theorem concerning the configurations symmetric to two given ones.

Theorem 3 *The workspace $\mathcal{W}$ of $\mathcal{A}$ is assumed to be bounded, and $\delta_1 > 0$ is a real constant. Let q_a and q_b be two configurations with different positions. If q_i is a configuration symmetric to q_a and q_b simultaneously, there exists $\delta_i > 0$ such that:*

$$\forall q_a{}', q_b{}' \in \mathcal{C}_{free}, \quad (r(q_a{}', q_a) < \delta_i) \wedge (r(q_b{}', q_b) < \delta_i) \Rightarrow$$
$$\exists q_i' \in \mathcal{C}_{free} \ / \ (r(q_i', q_i) < \delta_1) \wedge$$
$$(q_i' \text{ configuration symmetric to } q_a{}' \text{ and } q_b{}')$$

Proof: Let P_a and P_b be the positions of the respective configurations q_a and q_b. The frame is set to $\mathcal{R}_{ab}$, which is centered in the middle I of the segment $[P_a P_b]$ and whose x-axis is oriented along the vector $\overrightarrow{P_a P_b}$. In this frame, the coordinates of q_a and q_b are respectively $(-r/2, 0, \alpha)$ and $(r/2, 0, \alpha')$. If $\beta = (\alpha' - \alpha)/2$, r and β are similar to those defined in Theorem 1 and in the proof of Theorem 2 (even if the configurations q_a and q_b are not symmetric). The set of positions of configurations simultaneously symmetric to q_a and q_b is a curve containing P_a and P_b having constant curvature $\kappa = 2\sin\beta/r$: r is not null, so $\kappa \in \mathbb{R}$ and the set is a line or a circle. This is shown in [7], proposition 6 (these symmetric configurations are called *split postures* in this article). The set of the symmetric configurations can therefore be defined if $\kappa \neq 0$ as (s refers to the length along the set of positions):

$$q_i(\kappa, s) \begin{cases} x(\kappa, s) = \sin(\kappa s)/\kappa \\ y(\kappa, s) = (\cos\beta - \cos(\kappa s))/\kappa \\ \theta(\kappa, s) = \kappa s - (\alpha + \alpha')/2 \end{cases}$$

and, if $\kappa = 0$ ($\iff \beta = 0 \iff \alpha = \alpha'$), as:

$$q_i(0, s) \begin{cases} x(0, s) = s \\ y(0, s) = 0 \\ \theta(0, s) = -(\alpha + \alpha')/2 = -\alpha \end{cases}$$

Using Taylor polynomials, we can prove that the function giving the configuration $q_i(\kappa, s)$ in $\mathbb{R}^3$ for each (κ, s) in $\mathbb{R}^2$, is continuously differentiable along $\mathbb{R}^2$.

Considering the configurations $q_a{}'$ and $q_b{}'$, we can define similar positions P_a' and P_b', a similar frame $\mathcal{R}_{a'b'}$ and similar notations r' and β'. If $\delta_i < r/3$, as $|r' - r| < 2\delta_i$, we have $r' > r/3 > 0$ and the set of configurations symmetric to $q_a{}'$ and $q_b{}'$ respects similar equations in the frame $\mathcal{R}_{a'b'}$. Moreover, the partial derivates of the function which associates the curvature of this set to (β, r) or (β', r') can be bounded on the intervals of β and r concerned. Let C_1 and C_2 be these bounds, we then can write $|\kappa' - \kappa| \leq |\beta' - \beta| + C_2|r' - r| \leq (C_1 + 2C_2)\,\delta_i \leq M_\kappa\,\delta_i$. It is important to note that C_1 and C_2, and therefore M_κ tends toward infinity when r approaches zero.

Remember q_a, q_b and $q_i(\kappa, s)$ are given (so are κ and s). Let D_x be the bound of the partial derivation along κ of the function $x(\kappa, s)$, on an interval including the values of κ and κ'. Then, as $|s' - s| \leq \delta_i$, we have $|x'(\kappa', s') - x(\kappa, s)| \leq D_x|\kappa' - \kappa| + |s' - s| \leq (D_x M_\kappa + 1)\,\delta_i$. It is possible to find a similar inequality for the y-coordinates, replacing the constant D_x by a constant D_y. If the center of the frames are I and I', and the orientations of their x-axis are ϕ and ϕ', the distance between the configurations $q_i(\kappa, s)$ and $q_i'(\kappa', s')$ verifies $r(q_i'(\kappa', s'), q_i(\kappa, s)) \leq |x'(\kappa', s') - x(\kappa, s)| + |y'(\kappa', s') - y(\kappa, s)| + |\theta'(\kappa', s') - \theta(\kappa, s)| + II' + (2M+1)|\phi' - \phi|$. As the inequalities $II' \leq \delta_i$ and $|\phi' - \phi| \leq 2\delta_i/r$ holds, it is possible to conclude that $r(q_i'(\kappa', s'), q_i(\kappa, s)) \leq M_d\,\delta_i$, assuming that $|s' - s| \leq \delta_i$.

To conclude, the implication of Theorem 3 is verified when $0 < \delta_i < \min(r/3, \delta_1/M_d)$. ∎

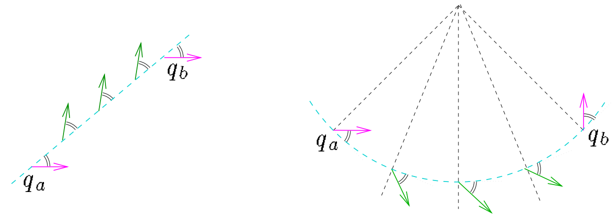

Figure 6: configurations symmetric to both q_a and q_b.

Then, we can deduce the important property:

Theorem 4 *The workspace $\mathcal{W}$ of $\mathcal{A}$ is assumed to be bounded, and $\varepsilon > 0$ is a real constant. Let q_a and q_b be two configurations linked by a non-extreme bi-elementary path Φ included in $\mathcal{C}^{\varepsilon}_{free}$. Then, there exists $\delta_2 > 0$ such that:*

$$\forall q_a{}', q_b{}' \in \mathcal{C}_{free}, \quad (r(q_a{}', q_a) < \delta_2) \wedge (r(q_b{}', q_b) < \delta_2)$$
$$\Rightarrow q_b{}' \in \mathcal{RC}_2(q_a{}', \mathcal{C}_{free})$$

Proof: Let Ψ^α and Ψ^ω be the first and second elementary path of Φ respectively, and q_i the configuration which ends Ψ^α and starts Ψ^ω. As Φ is not extreme, neither are Ψ^α or Ψ^ω. Thus, Theorem 2 can be applied to the configuration pairs (q_a, q_i) and (q_i, q_b), fixing $\delta_1{}^\alpha > 0$ and $\delta_1{}^\omega > 0$.

If $\delta_1 = \min(\delta_1{}^\alpha, \delta_1{}^\omega)$, we can apply Theorem 3 and find δ_i and q_i'. Setting $\delta_2 = \min(\delta_1, \delta_i)$ finishes the proof. ∎

To conclude, bi-elementary paths are interesting curves as they offer infinite possibilities to link almost any couple of configurations (if there is no obstacle; this property is stated more precisely and is proven in the report [14]). This property can be used to build a simplified planner returning a bi-elementary path, and a global path planner using this simplified planner (some examples of this kind of path planner are given in articles [10, 9, 15]). Theorem 4 can be used to prove the completeness of these path planners, assuming they satisfy a few properties. We will now give examples of this possibility with two planners, using the Ariadne's Clew algorithm [10] and the Probabilistic Path Planner method [15].

4 The Simplified Planner

Given two configurations q_a and q_b, the simplified planner searches a solution path linking q_a to q_b but, to avoid the (high) complexity of this task, it only checks for a limited set of paths, namely the set of the bi-elementary paths. Hence, our simplified planner has to find a configuration q_i symmetric to both q_a and q_b, such that the two elementary paths linking q_a to q_i and q_i to q_b are feasible (i.e. they exist and respect the turning radius constraint (2)) and collision-free. The search of this configuration q_i along the set of configurations symmetric to both q_a and q_b (this set is described in [7], proposition 6) can be executed using technics as various as dichotomy or stochastic method.

5 The CCPP planner

We will quickly describe our implementation of the Ariadne's Clew algorithm before proving its resolution-completeness (with respect to the set of paths considered).

5.1 Outline of the algorithm

From a general point of view, the Ariadne's Clew algorithm consists of exploring the configuration space from a starting configuration, as long as a simplified planner cannot link the explored region to the goal configuration and the exploration can continue. This algorithm is presented in article [10], and it is detailed in thesis [1].

The simplified planner corresponds to a function called SEARCH. The exploration of the configuration space is an approximation of the region of this space which can be reached from the starting configuration q_s with a solution path. This approximation is obtained by incrementally building a tree Λ of *landmarks*. The landmarks of this tree are configurations of $\mathcal{C}$ (the root of Λ is set to q_s), and the edges are solution paths: the tree gives a solution path linking q_s to any of its landmarks. To increase this tree, a function EXPLORE is used to find a new landmark λ as far as possible from the existing ones (those of the tree Λ). Therefore, the function EXPLORE has to solve an optimization problem. A stochastic method, called a *genetic algorithm* (a description of genetic algorithms and a review of their use in optimization can be found in the book [5]), is used to solve this optimization problem. In the CCPP planner, the function EXPLORE limits the paths searched to the set of feasible and non-colliding bi-elementary paths starting from the landmarks of Λ.

The exploration is stopped when the goal configuration can be reached or when the exploration is finished. Formally, this means that, if the new landmark λ can be linked to the goal configuration q_g using the simplified planner, the algorithm succeeds. If it cannot, and if this landmark is not far enough from the tree Λ, the algorithm suppose that Λ represents the region of $\mathcal{C}$ reachable from q_s, and that q_g is not in this region: CCPP fails.

The Ariadne's Clew Algorithm:

Initialization : $\Lambda = \{q_s\}, \lambda = q_s$, fail = false;
While (SEARCH does not find a path from λ to q_g) **and** (fail is false), **do**
  λ = EXPLORE;
  fail = $(\mathrm{d}(\lambda, \Lambda) < \delta)$
  $\Lambda = \Lambda \cup \{\lambda\}$;
End of While;
Conclusion : If fail
  then no solution found by the algorithm,
  else return the path from q_s to q_g, using Λ's tree structure and the result of SEARCH.

This planner is presented more precisely in article [12], along with experimental results.

5.2 Resolution-Completeness of CCPP

Using Theorem 4, we now can prove the resolution-completeness of CCPP.

Theorem 5 (Resolution-Completeness of CCPP) *The workspace $\mathcal{W}$ of $\mathcal{A}$ is assumed to be bounded, and $\varepsilon > 0$ is a real constant. Let Π be a path linking a configuration q_s to a configuration q_g.*

If Π is a sequence of non-extreme elementary paths included in $\mathcal{C}^{\varepsilon}_{free}$, there exists $\delta > 0$ such that CCPP finds, with the resolution δ, a path linking q_s to q_g.

Proof: Let $(\Pi_i)_{i\in\{1,\ldots,N\}}$ be the sequence of elementary paths of Π, and q_{i-1} and q_i the start and final configurations of the i^{th} elementary path Π_i of Π ($q_0 = q_s$ and $q_N = q_g$). Theorem 4 is applied to each of the bi-elementary paths formed with Π_i and Π_{i+1}, for $i \in \{1, \ldots, N-1\}$, and δ is set to the minimum of the $(N-1)$ δ_2 obtained. It can be proven that:

Lemma 1 *Under the hypotheses of Theorem 5, if CCPP does not find with the resolution δ a path linking q_s to q_g, there exists a landmark of Λ closer than δ from each configuration q_i, $i \in \{0, \ldots, N\}$.*

Sketch of the proof: This is proved by induction with a step 2 (it is true for 0 and 1, and if it is true for i, it is true for $i+2$). The initializations and the step use a *reductio ad absurdum* based on Theorem 4 and the failure of the planner. □

The resolution-completeness is thus proven by reduction to the absurd. If CCPP does not find a solution with the resolution δ, Lemma 1 allows us to say that:

$$\exists\, b_{N-2} \in \Lambda \;/\; r(b_{N-2}, q_{N-2}) < \delta$$

As the bi-elementary path formed with Π_{N-1} and Π_N respects the conditions of Theorem 4, the simplified planner of CCPP should have found a path linking b_{N-2} to q_g. This is in contradiction with the failure of CCPP. ■

6 Probabilistic Path Planning

The simplified planner can be used to define a probabilistic path planner with either a single robot or with several [15]. In this article, we will only briefly recall the single-robot method and prove its completeness (with respect to the set of paths considered).

Similarly to the Ariadne's Clew algorithm, the Probabilistic Path Planner (PPP) explores the configuration space in search of a solution path. The difference is that this exploration is based on the construction of a directed graph (instead of a tree), and is independent of the start and goal configurations, q_s and q_g (instead of being only independent of q_g).

For these two reason, this exploration can be done as a pre-treatment (it only depends on the environment), and multiple planning (or *queries*) can be executed using its resulting graph. The exploration is done by incrementally adding a random configuration q of $\mathcal{C}_{\text{free}}$ to the graph, and by trying to connect this configuration to a number of nodes of the graph, with the simplified planner. Each query consists in connecting the start configuration q_s and the goal one q_g to nodes of (the same connected component of) the graph, using the simplified planner, and then performing a graph search between the resulting nodes. The path deduced from the graph search is then smoothed to reduce its length (this can still be performed using the simplified planner).

This planner, using the simplified planner, is probabilistically complete with respect to the set of paths considered. This means that any problem which can be solved using paths made of elementary paths will be solved provided that the exploration is carried out for a sufficient amount of time:

Theorem 6 (Probabilistic Completeness of PPP) *The workspace $\mathcal{W}$ of $\mathcal{A}$ is assumed to be bounded, let $\varepsilon > 0$ be a real constant, and Π be a path linking a configuration q_s to a configuration q_g.*

If Π is a sequence of non-extreme elementary paths included in $\mathcal{C}^{\varepsilon}_{free}$, the planner PPP using our simplified planner will find a path linking q_s to q_g, as long as the exploration has been carried out for a sufficient amount of time.

Proof: The proof of this property is similar to the demonstration of [16] (Theorem 4) or to the demonstration of the completeness of CCPP (Theorem 5).

Taking the same notation that in this later proof, we have a property similar to Lemma 1: after a sufficient amount of time, the graph constructed by PPP will contain a node in each ball (in the classic metric of $\mathcal{C}$) of radius δ and center q_i. Using Theorem 4 and the bi-elementary paths of Π, we can prove that these nodes are in the same connected component of the graph. We can also prove that q_s and q_g can be connected to the graph. Therefore, PPP will find a path linking q_s to q_g. ■

7 Conclusion and Future Works

This paper presented *bi-elementary paths*, which are smooth and feasible paths for a car-like robot with a continuous curvature profile. It means that, in contrast to paths made of line segments and circular arcs (as usually produced by path planners), these paths can be followed by a real vehicle without stopping or diverging. A bi-elementary path is a sequence of two *elementary paths*, which are themselves composed of two symmetric arcs of clothoid (a clothoid is a curve whose curvature is a linear function of its arc length). These bi-elementary paths were used to define a simplified, i.e. non complete, planner which, in turn, has been used in two global planning schemes, namely the Ariadne's Clew algorithm and the Probabilistic Path Planning. In this paper, an important property of these paths has been shown, from which the completeness of the two global path planners has been deduced.

Bi-elementary paths are a first step in continuous-curvature path planning for car-like vehicles. Indeed, it can be shown that these paths are not optimal in length. This has led us to define new continuous-curvature paths made up of straight segments, clothoid arcs and circular arcs. First results concerning these new paths will hoppefully be found in [13].

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