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# Continuous-Curvature Trajectory Planning for Manoeuvrable Non-Holonomic Robots

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## Abstract

This article deals with trajectory planning for non-holonomic mobile robots. We call *trajectory* the association of a *path*, which is the geometrical restriction of the motion, and of a time schedule given by a time table, a velocity or an acceleration profile. The robots considered are non-holonomic, as their motion's direction is limited by the robot's position. However, two types of robot are considered: *car-like* vehicles have a lower bounded turning radius, while *manoeuvrable* robots do not. This article shows that optimal paths for car-like vehicles correspond to optimal trajectories for manoeuvrable robots, and presents how these paths can thus be followed by manoeuvrable robots faster than paths made of line segments, along which these robots have to stop at each direction's discontinuity. Experimental results obtained using such paths for manoeuvrable robots are then provided.

**Keywords** — mobile-robot, trajectory-planning, non-holonomic-system, optimality.

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## Abstract

*This article deals with trajectory planning for non-holonomic mobile robots. We call* trajectory *the association of a* path, *which is the geometrical restriction of the motion, and of a time schedule given by a time table, a velocity or an acceleration profile. The robots considered are non-holonomic, as their motion's direction is limited by the robot's position. However, two types of robot are considered:* car-like *vehicles have a lower bounded turning radius, while* manoeuvrable *robots do not. This article shows that optimal paths for car-like vehicles correspond to optimal trajectories for manoeuvrable robots, and presents how these paths can thus be followed by manoeuvrable robots faster than paths made of line segments, along which these robots have to stop at each direction's discontinuity. Experimental results obtained using such paths for manoeuvrable robots are then provided.*

## 1 Introduction

The first motion planning problem considered in mobile robotics was the "piano mover" problem: a two dimensional robot moving without constraints (it translates and rotates freely) among obstacles. Once this purely geometrical problem has been solved, new problems have been considered, adding kinematic (*i.e.* purely geometrical) then dynamic (*i.e.* related to time) constraints to the motion of the robot. The addition of these constraints reflects the complexity of the robot's model, increases the complexity of the planning but generally reduces the mechanical complexity of the robot.

Nowadays robot models are mainly of two kinds: the *car-like vehicle* model, which has a bounded turning radius, and the *manoeuvrable mobile robot* model, without bounded turning radius. Both kind of mobile robots can only move in a direction fixed by their position (they cannot translate freely), but the manoeuvrable ones can turn without moving (it rotates freely). The **main contribution** of this paper is to prove that trajectory planning for manoeuvrable mobile robots can use, when high velocities are required (*i.e.* when dynamic aspects must be taken into account), optimal paths for car-like vehicles: the same algorithms can then be used to find a solution.

### Overview of this article

A short bibliographic section recalls the main researches considering path and trajectory planning for mobile robot (§ 2). While the existence of constraints similar to those for car-like vehicles has already been proved for a first model of manoeuvrable mobile robot, a robot corresponding to a second model is presented and the related planning problem is stated (§ 3). Once this problem is characterized in terms of existence and optimality of solutions, a method to compute optimal trajectories is described (§ 4) and experimental results are given (§ 5). At last, a conclusion and future works are sketched (§ 6).

## 2 Related Works

Let us define three kinds of mobile robots' model:

1. *Unconstrained robots* can rotate and translate freely (the planning problem is then called the "piano mover" problem);
2. *Manoeuvrable robots* can rotate freely (unbounded turning radius) but can only translate in a direction fixed by its position;
3. *Car-like robots* cannot rotate nor translate freely, its turning radius is lower bounded and its translation's direction is fixed by its position.

These models are presented in increasing order of the planning complexity, which is also the chronological order in which they have been considered.

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Indeed, paths planned for unconstrained or manoeuvrable robots are usually only made of line segments while for car-like robots these segments are tangentially connected by circular arcs of minimum radius [7]. These paths have optimal length: this is an evidence in the case of unconstrained or manoeuvrable robots, and has been proved by Dubins for car-like robots going only forward [2] and by Reeds and Shepp for car-like robots going both forward and backward [12]. Unfortunately, in any case, these paths cannot be followed precisely without stopping at each discontinuity (of the direction in the first case, of the turning radius in the second) to reorient the robot's directing wheels.

To avoid these stops, the use of continuous-curvature paths has been recommended for more than ten years. Various kinds of continuous-curvature paths have been used, as *e.g.* clothoids[1] [5], cubic spirals[2] [4], B-splines [6] or Cartesian and polar polynomials [10, 16]. However, if some of these paths correspond to optimality criteria [4, 16], these criteria are not based on a robot's model and their interest are not compared: it is thus difficult to select which type of paths is the more interesting. Moreover, we only found one of these works generating continuous-curvature paths for car-like vehicles [9].

To the best of our knowledge, only two works considered continuity of the curvature as a constraint added to the motion planning problem, and searched optimal paths for the new planning problem. The first one [8] is designed for a manoeuvrable robot called *Hilare 2*, while the second [14] focussed on car-like vehicles. The paths generated are made of pieces of clothoid and anti-clothoid[3] in the first case, and made of pieces of clothoid (including line segments) and of circular arcs of minimum radius in the second case.

In this paper, we show why the path used in the second case are interesting for manoeuvrable robot when high velocities are expected. In the Hilare 2 case, a dynamic non-sliding constraint on each wheel induces a lower-bound on the turning radius, and thus add circular arcs to the parts of optimal paths. We will show that similar paths (*cf.* § 3.3) are needed for another type of manoeuvrable robot: the Nomad 200™.

# 3 Planning Problem

Before stating the planning problem (§ 3.2), we will shortly present the Nomad 200™ and the model we used for it (§ 3.1). We will then characterize the problem w.r.t. existence and nature of optimal trajectories (§ 3.3).

[1] A clothoid is a curve whose curvature is a linear function of the arc length.

[2] A "cubic spiral" is a curve whose curvature is a quadratic function of the arc length.

[3] Anti-clothoids, or involutes of circles, as defined by Fleury *et al.* in [3], are curves whose radius is a linear function of time, while its steering velocity is constant.

## 3.1 Model of the Robot

Built by Nomadic Inc., the Nomad 200™ is a roughly cylindrical mobile robot (*cf.* Fig. 1), mounted on three wheels. The lower part of the robot keeps a fixed direction while the three wheels it contains can turn in any direction, but remain parallel. Thus, the turning radius of this robot is not lower bounded: the robot is highly maneuverable.

Figure 1: the Nomad 200™.

As the Nomad 200™ will only move on a planar horizontal ground, the workspace $\mathcal{W}$ is represented by a bounded closed subset of the plane $\mathbb{R}^2$. In this workspace, we model a position of the Nomad 200™ by an oriented circle, noted $\mathcal{A}$, of fixed radius $R_{\mathcal{A}}$ (*cf.* Fig. 2). It is represented by a three dimensional vector (a *geometric configuration*) $(x, y, \theta) \in \mathbb{R}^2 \times \mathcal{S}^1$, the two first coordinates giving the position of the circle's center and the third being the orientation of the oriented circle (*i.e.* the angle between the $x$-axis and the wheels' direction).

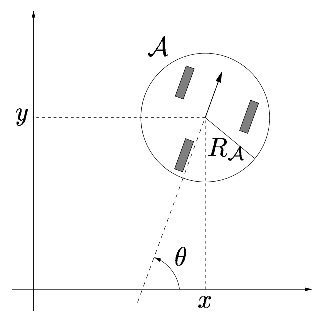


Figure 2: Geometric Configuration of the Nomad 200™.

As we want the robot's wheels to roll without sliding, its motion has to respect a non-holonomic constraint: at a given geometric configuration, its instantaneous velocity is always parallel to its wheels' direction. Using the notations previously defined, this constraint can be written in the workspace $\mathcal{W}$ as:

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \qquad (1)$$

We are also interested in the robot's dynamics, and thus we have to add the translation and steering velocity to the configuration's parameters. Let us call *dynamic configuration* the combination of the geometric configuration's parameters and of its derivatives (w.r.t. time). A dynamic configuration of $\mathcal{A}$ is thus a 5 dimensional vector $(x, y, \theta, v, \omega) \in \mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}^2$, where $x$ and $y$ are the coordinates of the robot's vertical symmetry axis, $\theta$ is the orientation of its wheels, $v$ is its translation velocity and $\omega$ is its steering velocity.

The non-sliding constraint (1) can then be rewritten in the dynamic configuration space as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ \theta \\ v \\ \omega \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ a \\ \gamma \end{pmatrix} \tag{2}$$

where $a$ and $\gamma$ are respectively the translation and steering accelerations, and are the control parameters. The robot has to respect an other constraint, giving its dynamic limitations (*i.e.* its maximum velocities and accelerations):

$$\left\{ \begin{array}{lcl} |v| & \leq & v_{\max} \\ |\omega| & \leq & \omega_{\max} \\ |a| & \leq & a_{\max} \\ |\gamma| & \leq & \gamma_{\max} \end{array} \right. \tag{3}$$

## 3.2 Statement of the Problem

We just saw that the motions of our robot $\mathcal{A}$ are limited by non-holonomic (*i.e.* non-integrable) constraints. Thus, to move from a configuration to an other one while avoiding obstacles, $\mathcal{A}$ can only follow a restricted set of trajectories. It will then need a *planner* to find among these trajectories the optimal one.

To define formally the *planning problem* we need to solve, let us denote $\mathcal{C}$ the configuration space, *i.e.* the subset of $\mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}^2$ containing all the dynamic configurations for which $\mathcal{A}$ is in the workspace $\mathcal{W}$. If $q$ is a configuration in $\mathcal{C}$, let $\mathcal{A}(q)$ be the (circular) region occupied in $\mathcal{W}$ by $\mathcal{A}$ when it is in configuration $q$. Then, if $\mathcal{B}$ is an obstacle in $\mathcal{W}$, we say that $q$ is *in collision* with $\mathcal{B}$ if, and only if, $\mathcal{A}(q) \cap \mathcal{B} \neq \emptyset$.

If the workspace $\mathcal{W}$ is clustered with a set of obstacles $\mathcal{B}_j$, $j \in \{1, ..., n_\mathcal{B}\}$, let us denote $\mathcal{C}_{\text{collision}}$ the set of all the configurations of $\mathcal{C}$ that are in collision with an obstacle $\mathcal{B}_j$, $j \in \{1, ..., n_\mathcal{B}\}$, and let $\mathcal{C}_{\text{free}}$ be its complement:

$$\begin{array}{rcl} \mathcal{C}_{\text{collision}} & = & \{q \in \mathcal{C} \ / \ \exists\, j \in \{1, ..., n_\mathcal{B}\}, \mathcal{A}(q) \cap \mathcal{B}_j \neq \emptyset\} \\ \mathcal{C}_{\text{free}} & = & \mathcal{C} \setminus \mathcal{C}_{\text{collision}} \\ & = & \{q \in \mathcal{C} \ / \ \forall\, j \in \{1, ..., n_\mathcal{B}\}, \mathcal{A}(q) \cap \mathcal{B}_j = \emptyset\} \end{array}$$

Moreover, if $\varepsilon$ is a positive real value, $\mathcal{C}^{\varepsilon}_{\text{free}}$ is the set of all the configurations $q$ of $\mathcal{C}$ such that the distance in $\mathcal{W}$ between $\mathcal{A}(q)$ and any obstacle $\mathcal{B}_j$, $j \in \{1, ..., n_\mathcal{B}\}$, is greater than (or equal to) $\varepsilon$.

A trajectory is then a continuous curve in $\mathcal{C}$, and a *collision-free trajectory* is a continuous curve in $\mathcal{C}_{\text{free}}$. Such a trajectory is said *feasible* for $\mathcal{A}$ if, and only if, it respects the motion's constraints of this robot, *i.e.* if, and only if, it respects Equations (2) and (3).

A formal statement of the planning problem we consider is then the following:

Being given a starting configuration $q_s$, a goal configuration $q_g$, the set of obstacles $\mathcal{B}_j$, $j \in \{1, ..., n_\mathcal{B}\}$, and a real $\varepsilon > 0$, we search a trajectory $\Gamma$ which:

- connects $q_s$ to $q_g$ ($\Gamma$ is a curve in $\mathcal{C}$);
- is feasible for $\mathcal{A}$, *i.e.* respects Equations (2) and (3);
- is a curve in $\mathcal{C}^{\varepsilon}_{\text{free}}$, *i.e.* such that the distance between $\mathcal{A}$ and any obstacle remains greater than $\varepsilon$.

**Remarks:**

- $\varepsilon$ can be interpreted as a "security distance" to the obstacles; the fact that the search is limited to $\mathcal{C}^{\varepsilon}_{\text{free}}$ (instead of only $\mathcal{C}_{\text{free}}$) will also ensure of the existence of a time-optimal solution (whenever a solution exists, see § 3.3); indeed, due to equation (3), $\mathcal{C}$ is now a subset of $\mathcal{W} \times \mathcal{S}^1 \times [-v_{\max}, v_{\max}] \times [-\omega_{\max}, \omega_{\max}]$ and is a compact set, as well as $\mathcal{C}^{\varepsilon}_{\text{free}}$ for all $\varepsilon > 0$, while $\mathcal{C}_{\text{free}}$ is an open set;
- the search for a solution trajectory is done in $\mathcal{C}$ instead of in $\mathcal{C}^{\varepsilon}_{\text{free}}$ (the projection of the obstacles $\mathcal{B}_j$, $j \in \{1, ..., n_\mathcal{B}\}$, in the configuration space $\mathcal{C}$ is too complex to be computed), the collision avoidance being verified in $\mathcal{W}$.

## 3.3 Controllability and Optimality

Now that the planning problem we consider is formally stated, let us study some of its properties.

First of all, the robot $\mathcal{A}$ we modeled can move as a holonomic robot: $\mathcal{A}$ can turn without moving, and it can follow straight lines. It means that a solution trajectory exists if, and only if, $q_s$ and $q_g$ are in the same connected part of $\mathcal{C}^{\varepsilon}_{\text{free}}$. However, the simplest trajectories (made of pure rotations and of straight line motions) are not *optimal* (*i.e.* the shortest in time).

In our case, when a solution trajectory exists, we look for the trajectory whose execution time is the smallest. This can be considered as a Lagrange optimization problem [1, Chap. 5], for which the conditions of the Filippov Theorem adapted to this class of problems [1, Th. 5.1.ii or 9.3.i] are respected (for more details, refer to Appendix A). As a consequence, whenever a solution trajectory exists, a solution trajectory with minimum execution time can be found. This is the case as the search is limited to $\mathcal{C}^{\varepsilon}_{\text{free}}$, which is a compact set, instead of $\mathcal{C}_{\text{free}}$, which is open: indeed, would the search have been limited to $\mathcal{C}_{\text{free}}$, it would always have been possible to find a trajectory faster than any given one (simply by taking a trajectory closer to the obstacles).

To characterize the optimal solution trajectory, we used the Pontryagin Maximum Principle [11], which gives necessary conditions respected by the portions of the optimal trajectories included in the inner part of the search space (here the configuration space), *i.e.* in our case when neither the translation nor the steering velocity ($v$ and $\omega$) are maximum (in absolute value). In our case, this principle proves that both the translation and the steering acceleration (*i.e.* $a$ and $\gamma$) are *bang-bang* (*i.e.* zero or maximum in absolute value) along optimal trajectories, except when an obstacle's border is followed; details are given in Appendix B.

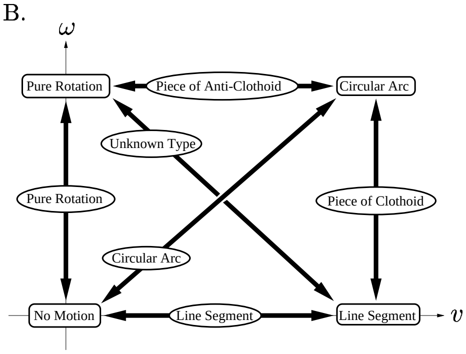


Figure 3: Optimal Control Curves.

Fig. 3 indicates the curves followed in the plane by the projection of the vertical symmetry axis of the robot $\mathcal{A}$, when control is optimal (as defined previously). In this figure, rounded boxes correspond to motions for which both (translation and steering) accelerations are zero (translation and steering velocities are constant), while ellipses correspond to motions with (at least) one maximum acceleration (in absolute value).

The nature of the optimal curves is computed in the following way:

- it is trivial when the translation velocity $v$ is zero;
- it is easily found considering that the curvature $\kappa$ along the curves is (if $s$ is the arc length):

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\omega}{v}$$

The line segments correspond to a zero curvature, the circular arcs to a constant (non zero) curvature, the clothoid pieces to a linear curvature (w.r.t. the time) with constant translation velocity and the anti-clothoid pieces to a linear radius of curvature (the inverse of the curvature) with a constant steering velocity (see Fig. 3).

The name of one of the optimal curves for the Nomad 200$^{\mathrm{TM}}$ has not yet been identified: when the translation velocity $v$ decreases (resp. increases) with maximum translation acceleration $a_{\max}$ (in absolute value) while the steering velocity $\omega$ increases (resp. decreases) with maximum steering acceleration $\gamma_{\max}$ (in absolute value), the coordinates of the curve followed by the robot $\mathcal{A}$ is a combination of the coordinates of a circle and of those of a clothoid. However, in the planning, we will only use the curves of the lower right half of Fig. 3, *i.e.* linear segments, circular arcs and pieces of clothoids, as we want the translation velocity $v$ to remain maximum as long as possible.

**Remark:** Optimal paths for the Nomad 200$^{\mathrm{TM}}$ include, but are not restricted to, optimal paths for the Hilare 2. The difference comes from the control parameters: the Nomad 200$^{\mathrm{TM}}$ is controlled by its translation and turning accelerations (resp. $a$ and $\gamma$), while the Hilare 2 is controlled

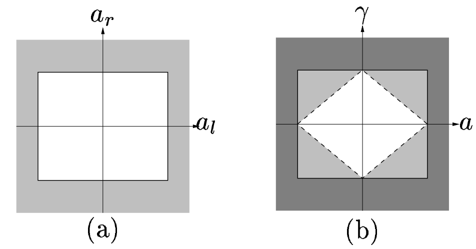


Figure 4: Control Regions (a- Hilare 2, b- Nomad 200$^{\mathrm{TM}}$).

by the accelerations of its left and right wheels (resp. $a_l$ and $a_r$). If these controls are equivalent (a bijection exists between them), standart bounds define different control regions (*cf.* Fig. 4). Indeed, a rectangular region in the $(a_l, a_r)$ space (used for the Hilare 2) is transformed in a diamond-shaped region in the $(a, \gamma)$ space (used for the Nomad 200$^{\mathrm{TM}}$). This difference between the allowed control regions explains the difference between the optimal paths.

# 4 Planning Method

We search for fast trajectories, *i.e.* optimal trajectories for which the translation velocity remains maximum as long as possible. In that case, paths (*i.e.* the geometric restrictions of the motion) will only be made of line segments, pieces of clothoids and circular arcs. A planner returning this kind of paths has already been defined for car-like vehicles [14]. We extended this planner to compute paths in our case.

As for the dynamic aspects of the motion, the velocity profile of the trajectory can be divided in at most three phases: an *acceleration phase* during which the translation velocity is maximally increased from its original value to the maximum value, a phase with maximum translation velocity and a *deceleration phase* during which the translation velocity is maximally decreased from its maximum value to its final value.

The resulting trajectory respects the robot's kinematic and dynamic constraints (as specified in § 3.1), avoids the set of obstacles and keeps translation velocity as high as possible. This does not prove that the resulting path is time optimal: we also consider whether the trajectory made of pure rotations (made with zero translation velocity) and pure translations is faster.

# 5 Experimental Results

Fig. 5 gives two example of planning for the Nomad 200$^{\mathrm{TM}}$, using the extension of the planner for car-like vehicles [14]. In these examples, the continuous-curvature trajectory with bounded curvature is much faster than a trajectory made of pure translations and pure rotations, as at least four stops would have been required in this case. It can be noticed that the continuous-curvature trajectories begin and end with a circular arc, along which both translation and turning accelerations (resp. $a$ and $\gamma$) are maximum.

These continuous-curvature trajectories should be followed very accurately, as they correspond to the con-

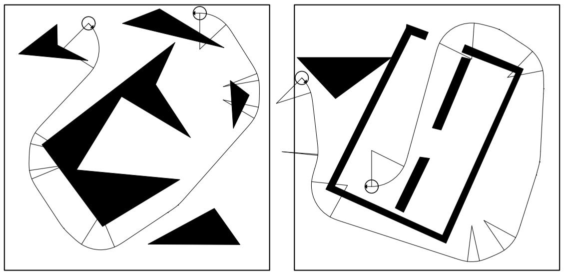

Figure 5: planning for the Nomad 200™.

troller's robot model. Moreover, the tracking of the corresponding paths is ten times better than the tracking of discontinuous-curvature paths, using a Kanayama law (*cf.* [15]). However, only simulated results have been obtained, as the controller developed by Nomadic Inc. for the Nomad 200™ is dramatically inaccurate: translation and turning accelerations are jerky and do not respect the fixed bounds. Future works are planned to improve this controller in order to be able to follow the planned trajectories.

# 6 Conclusion and Future Works

Usual mobile robots can only translate in a direction fixed by their position. Some of these robots have also a lower bounded turning radius. They are called *car-like* vehicles while those without bounded turning radius are called *manoeuvrable* robots.

This article shows how sub-optimal continuous-curvature paths for car-like vehicles can be used to define high velocity trajectories for manoeuvrable robots. A first case, considered by Laumond *et al.* [8], is quickly recalled. A second case is described more precisely: optimal trajectories are proved to use paths similar to those optimal for car-like vehicles, and thus a continuous-curvature path planner for car-like vehicles is used to generate smooth trajectories which can be followed with a high velocity. Experimental results, showing the tracking speed-up w.r.t. trajectories made of pure rotations and translations, are presented.

**Acknowledgments** — The work presented in this paper was realized during A. Scheuer's post-doctoral Research Fellowship, funded by ARC1/94 project's manpower budget, in the "Vision and Control" Strategic Research Program of the Nanyang Technological University, as a collaboration between this university and the French institute INRIA.

# Appendix

We proved that there exists a solution to the planning problem stated in § 3.2 if, and only if, the configurations to connect are in the same connected part of $\mathcal{C}^{\varepsilon}_{\text{free}}$. In this appendix, we will prove that, in that case, there also exists a time-optimal solution and we will show a few properties of this optimal solution.

First of all, let us recall the problem:

Being given a starting configuration $q_s$, a goal configuration $q_g$, and the set of obstacles $\mathcal{B}_j$, $j \in \{1, ..., n_{\mathcal{B}}\}$, we search a trajectory $\Gamma$ which:

- connects $q_s$ to $q_g$ ($\Gamma$ is a curve in $\mathcal{C}$);
- is feasible for $\mathcal{A}$, *i.e.* respects the equations (2) and (3);
- is a curve in $\mathcal{C}^{\varepsilon}_{\text{free}}$, *i.e.* such that the distance between $\mathcal{A}$ and any obstacle remains greater than $\varepsilon$.

The trajectory $\Gamma$ can be considered as a mapping from time to configurations, *i.e.* as a function $t \mapsto q(t)$ from $[0, T]$ to $\mathcal{C}^{\varepsilon}_{\text{free}}$. To find the time optimal solution trajectory, we want to optimize

$$I[\Gamma, u] = g(0, q(0), T, q(T)) + \int_0^T f_0(t, q(t), u(t))dt,$$

where $g = 0$, $f_0 = 1$ and the functions $q$ (the trajectory) and $u$ (its associated control function) verify:

- the differential system

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}t} &= f(t, q(t), u(t)) \\ &= (v(t)\cos\theta(t), v(t)\sin\theta(t), \omega(t), u_0(t), u_1(t))\end{aligned}$$

- the limit conditions

$$(0, q(0), T, q(T)) \in B = \{0\} \times \{q_s\} \times [0, T_{\max}] \times \{q_g\}$$

- and the constraints

$$\begin{cases} (t, q(t)) & \in \quad A = [0, T_{\max}] \times \mathcal{C}^{\varepsilon}_{\text{free}} \\ u(t) & \in \quad U = [-a_{\max}, a_{\max}] \times [-\gamma_{\max}, \gamma_{\max}] \end{cases}$$

# A Existence of Optimal Paths

Before giving Filippov's theorem, which proves the existence of time-optimal solutions to our problem, let us define, for all $(t, q)$ in $A$:

$$\begin{aligned}\widetilde{Q}(t, q) &= \left\{(z^0, z)/\exists u \in U, z^0 \geq f_0(t, q, u), z = f(t, q, u)\right\} \\ &= \left\{(z^0, v\cos\theta, v\sin\theta, \omega, u), z^0 \in [1, +\infty[, u \in U\right\}\end{aligned}$$

**Theorem 1 (Filippov)** *Let us assume that $A$ and $U$ are compact sets, $B$ is a closed set, $f_0$ and $f$ are continuous on $M = A \times U$, $g$ is continuous on $B$ and $\widetilde{Q}(t, q)$ is a convex set for all $(t, q)$ in $A$.*
*Then, if there exists solutions to the considered problem, $I[\Gamma, u]$ reaches an absolute minimum on the set of solutions.*

**Corollary 1** *If $q_s$ and $q_g$ are in the same connected part of $\mathcal{C}^{\varepsilon}_{\text{free}}$, there exists a time optimal solution to the planning problem we consider.*

*Proof:* It is easy to verify that $A$ and $U$ are compact sets (let us recall that $\mathcal{C}^{\varepsilon}_{\text{free}}$ is a compact set), that $B$ is a closed set, and that the functions $f_0$, $f$ and $g$ are continuous.

Moreover, $\widetilde{Q}(t, q)$ is a convex, as it is isomorphic to $[-a_{\max}, a_{\max}] \times [-\gamma_{\max}, \gamma_{\max}] \times [1, +\infty[\subset \mathbb{R}^3$ for all $(t, q)$ in $A$. □

# B Nature of Optimal Paths

To show some properties of the optimal trajectories solution of our problem, we will use the Pontryagin Maximum Principle as stated by Cesari [1, Chap. 4]. This principle can only be applied to the trajectories contained in the inner part of $\mathcal{C}^{\varepsilon}_{\text{free}}$. However, if the workspace $\mathcal{W}$ is chosen wide enough, the pieces of the optimal trajectories which are on the boundaries of $\mathcal{C}^{\varepsilon}_{\text{free}}$ either follow the borders of an obstacle (at a distance $\varepsilon$) or correspond to a maximum (in absolute value) translation or steering velocity. Nothing can be said in the first case (the nature of the trajectory depends on the geometric shape of the obstacle). The two other cases will be considered at the end of this appendix, once the Pontryagin Maximum Principle has been applied to the trajectories contained in the inner part of $\mathcal{C}^{\varepsilon}_{\text{free}}$.

Let us now consider the pieces of the optimal trajectories that are in the inner part of $\mathcal{C}^{\varepsilon}_{\text{free}}$. For those pieces, the necessary conditions of the Pontryagin Maximum Principle [1, Chap. 4, cond. (a)-(d)] are verified:

(a) the existence of an optimal trajectory has been proved using Filippov's theorem (*cf.* Appendix A);

(b) the curve $(t, q(t))$ corresponding to the concerned piece of optimal trajectory remains in the inner part of $A$;

(c) $U$ is a bounded and closed set of $\mathbb{R}^2$;

(d) $B$ has a linear tangent variety $B'$ whose vectors are $h = (0, 0, \tau, 0), \tau \in \mathbb{R}$ (the elements of $B$ have constant coordinates, except for the third which remains in a interval).

Application of the Pontryagin Maximum Principle then imply that either $a$ or $\gamma$ is maximum (in absolute value) on any interval (the detailed demonstration can be found in the appendix of [13]). Using once again the Pontryagin Maximum Principle on both reduced problems (with $q_a = (x, y, \theta, \omega)$ for $a(t) = \pm a_{\max}$, and $q_\gamma = (x, y, v)$ for $\gamma(t) = \pm\gamma_{\max}$) leads to the following conclusion : both $a$ and $\gamma$ are maximum (in absolute value) or the trajectory followed is a line segment.

Thus, we proved that the optimal trajectories contained in the inner part of $\mathcal{C}^{\varepsilon}_{\text{free}}$ are *bang-bang*, *i.e.* that they correspond to maximum or zero translation and steering accelerations. We still have to prove that it is also the case for the optimal trajectories contained on the boundary of $\mathcal{C}^{\varepsilon}_{\text{free}}$. As we already noticed, nothing can be said about the optimal trajectories which follow the borders of obstacles. However, the other ones correspond to a maximum (in absolute value) translation or steering velocity. In that case, we can prove that the accelerations are bang-bang (*i.e.* zero or maximum in absolute value): once again, it is done using the Pontryagin Maximum Principle on the reduced problems, with $q_v = (x, y, \theta, \omega)$ for $v = \pm v_{\max}$, and $q_\omega = (x, y, v)$ for $\omega = \pm\omega_{\max}$. The only possible exception is, when $v = \pm v_{\max}$, a line segment which correspond to $\omega = \gamma = 0$.

**As a conclusion**, optimal trajectories for our robot are achieved with *bang-bang* control, *i.e.* $a(t) = \pm a_{\max}$ except if $v(t) = \pm v_{\max}$ (then $a(t) = 0$) and $\gamma(t) = \pm\gamma_{\max}$ except if $\omega(t) \in \{-\omega_{\max}, 0, \omega_{\max}\}$ (then $\gamma(t) = 0$), when they do not follow an obstacle's boundary.

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