Causal Dynamics of Discrete Manifolds

Pablo Arrighi, Simon Martiel

U. of Grenoble, U. Nice Sophia Antipolis

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Outline

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Causal Graph Dynamics

Graphs and Oriented Complexes

Pachner moves and homeomorphisms

Motivation

Discretized time evolutions in physics (lattice-gas models, cellular automata...). Generalized:

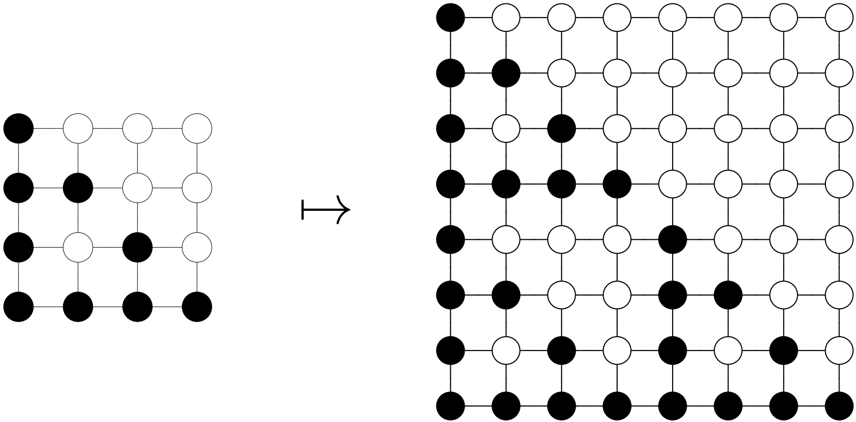
- To a general discrete space: Graphs
- Keeping the symmetries of physics: Causality and Translation-invariance

Two definitions for the same object:

- Axiomatic definition (physical/mathematical)
- Constructive definition (computational)

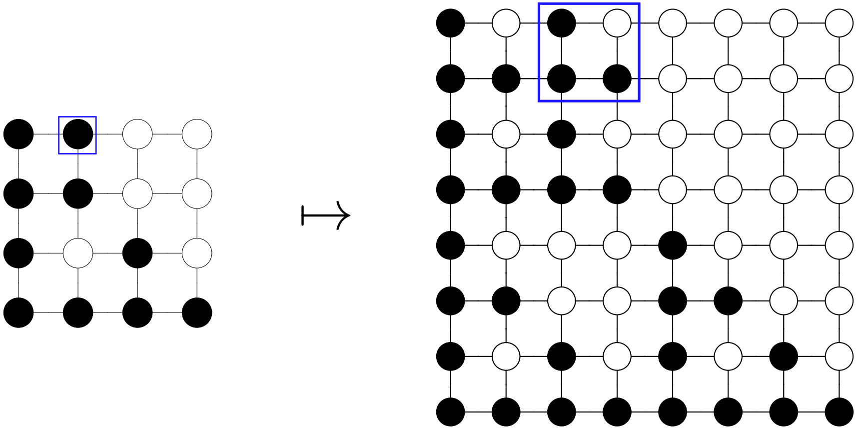
Axiomatic view

Evolutions of graphs



Axiomatic view

Evolutions of graphs
+ causality



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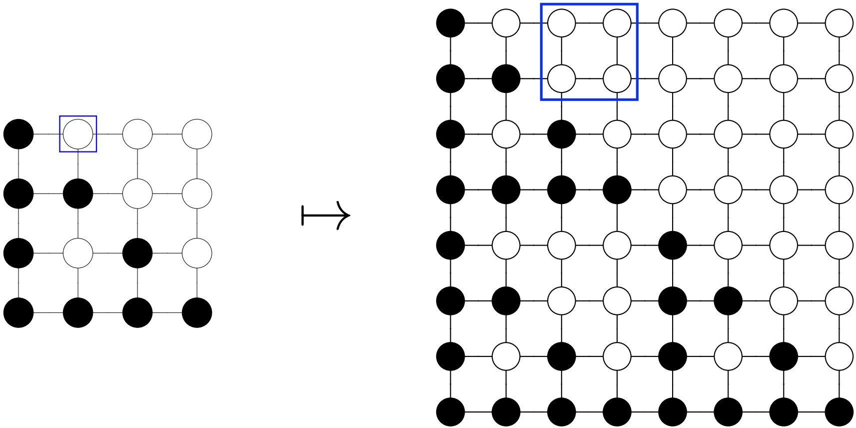
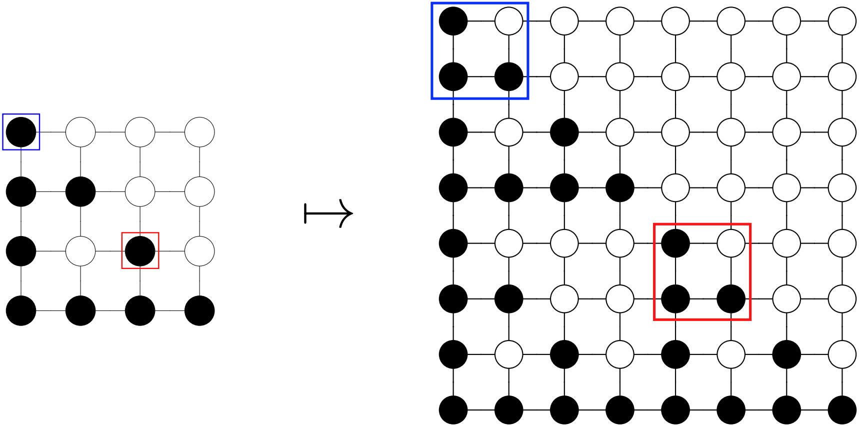
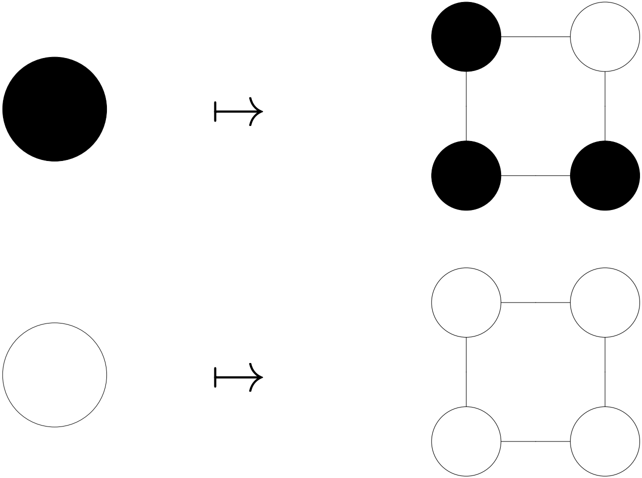


Axiomatic view

Evolutions of graphs
+ causality
+ translation invariance



Constructive view



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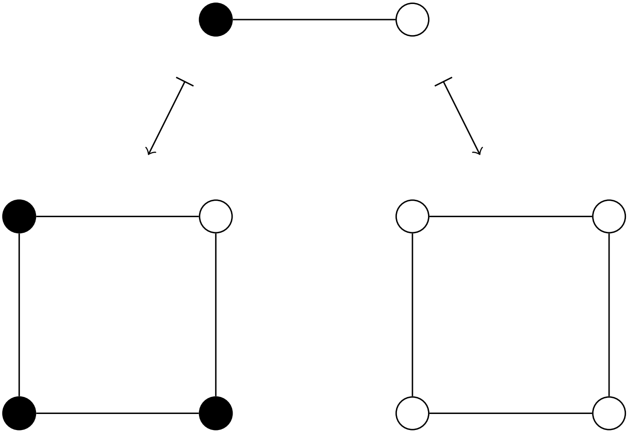
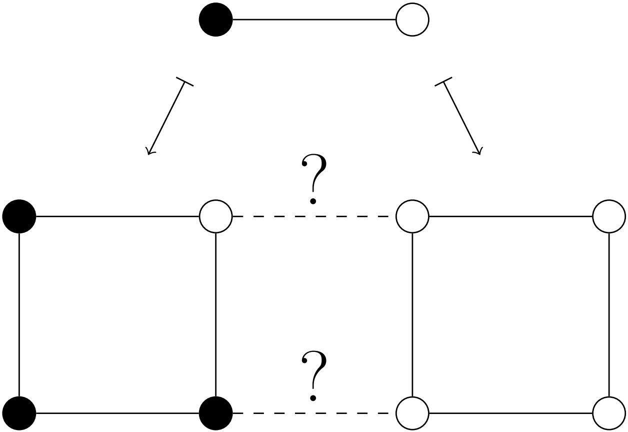


Constructive view

OK, but how to glue all the subgraphs together?



Constructive view

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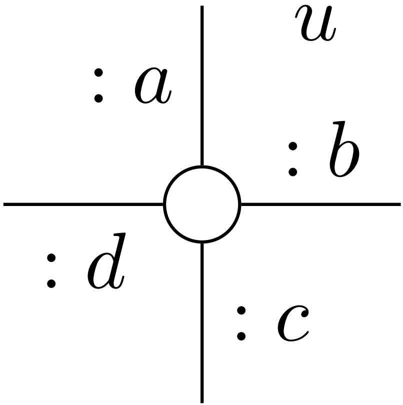

Constructive view

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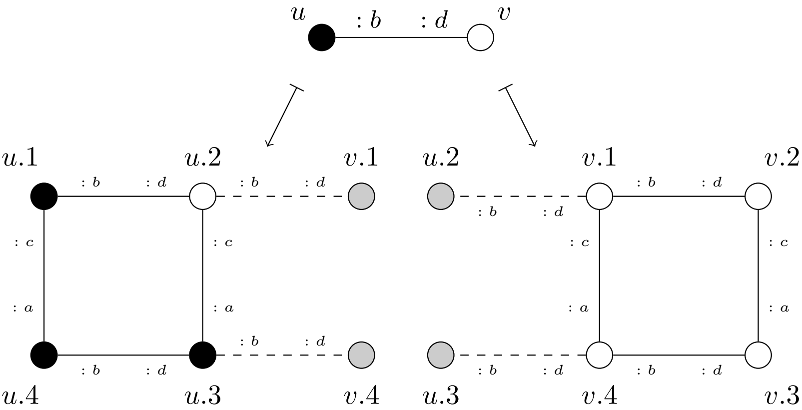
Have each vertex order its edges. Make the subgraphs overlap.



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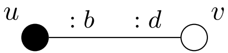
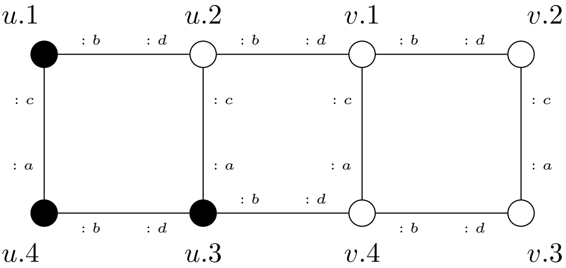




Some results on this generalization of CA

Axiomatic definition: (Causal dynamics)

- Pointed graphs endowed with a Cantor metric.
- Causality as continuous functions w.r.t. the metric.
- Translation invariance as a commutation with isomorphism.

Constructive definition: (Localizable dynamics)

- $F(G)$ induced by a local rule f.

Theorem [AD12a, AD12b][AM12]

The axiomatic definition equivalent to the constructive definition.

Theorem [AM12]

- Local rules f are enumerable
- The induced $G \mapsto F(G)$ is computable

Outline

Causal Graph Dynamics

Graphs and Oriented Complexes

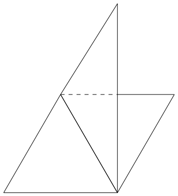
Pachner moves and homeomorphisms

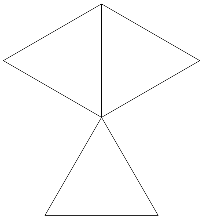
Characterize...

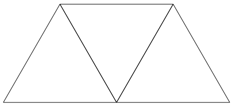
...pseudo-manifolds:

- Simplicial/Δ complexes.
- Obtained by glueing simplices on facets.

NO: NO: YES:







Correspondance between graphs and pseudo manifolds?

Consider n-dimensional pseudo-manifolds.

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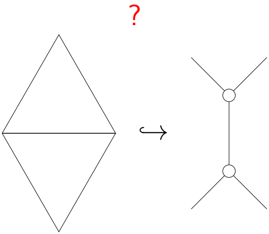
Is there an encoding taking:

- A simplex $\hookrightarrow$ A vertex.
- The glueing of two facets $\hookrightarrow$ A (labelled) edge.
- A pseudo manifold $\cong$ A (labelled) graph.



Correspondance between graphs and pseudo manifolds?

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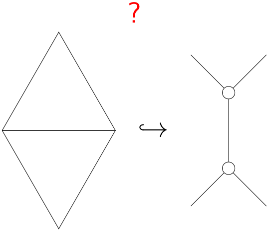
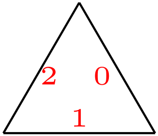


How much more is there to a simplicial complex than there is to a graph?

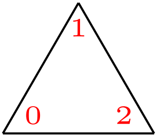
Colored Complexes

Number the $n+1$ faces of a n-simplex with $\{0, ..., n\}$.



Colored Complexes

Number the $n + 1$ vertices of a n-simplex with $\{0, \ldots, n\}$.



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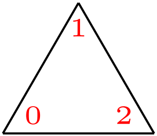
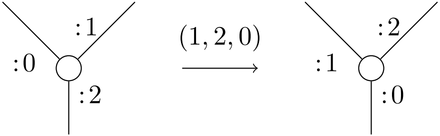


Close to our graphs, but still not an oriented complex

Vertex rotation and symmetry

A vertex rotation:



Formally:

- Vertex rotation: Even permutation of the ports
- Vertex symmetry: Odd permutation of the ports

Rotations define 2 orientations:

- On 2-simplices (triangles):

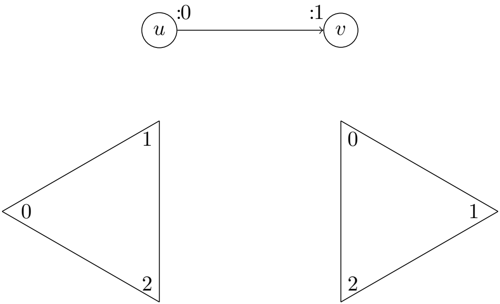
  Clockwise vs. Counter Clockwise
- On 3-simplices (tetrahedra):

  Three fingers rule: left hand vs. right hand.

Vertex modulo rotations ↔ oriented *n*-simplex.

Oriented glueing of n-simplices

In 2 dimensions:



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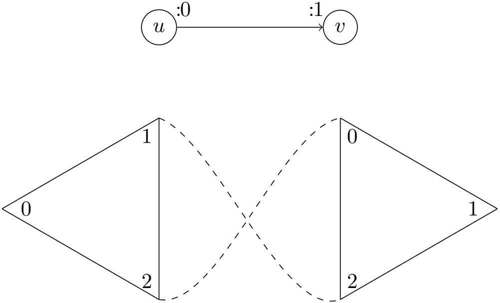

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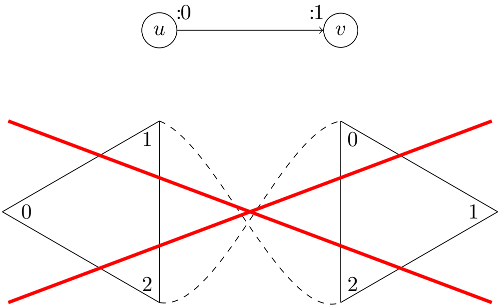

Oriented glueing of *n*-simplices

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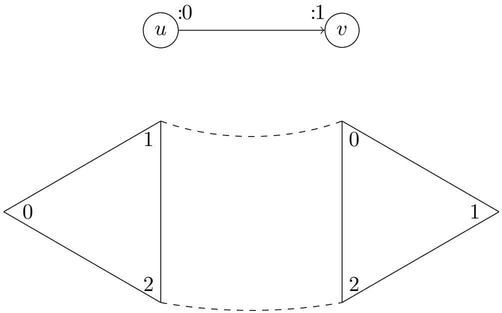
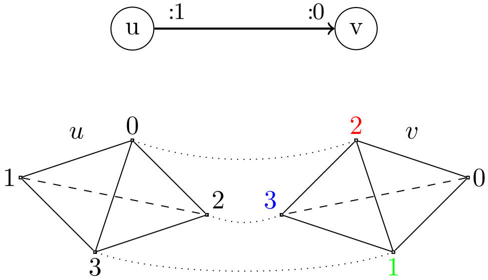


Oriented glueing of n-simplices

In 3 dimensions:



Oriented glueing of n-simplices

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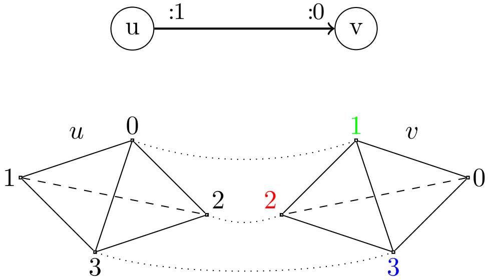

Oriented glueing of *n*-simplices

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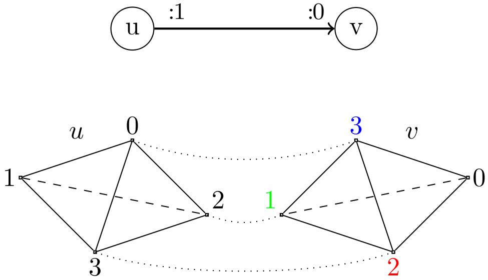
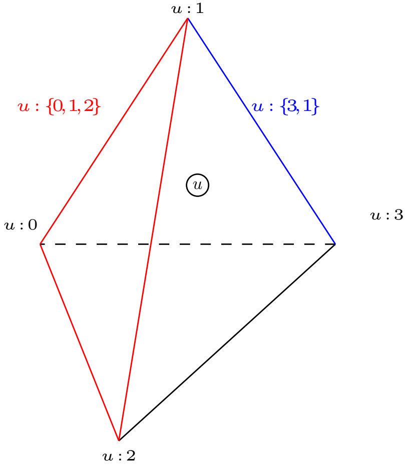


Oriented glueing of n-simplices

In n dimensions:

- $n!$ ways of glueing two n-simplices.
- $\frac{n!}{2}$ oriented ways.
- We can use an odd permutation to explicit the glueing.

Graph $\leftrightarrow$ Complex



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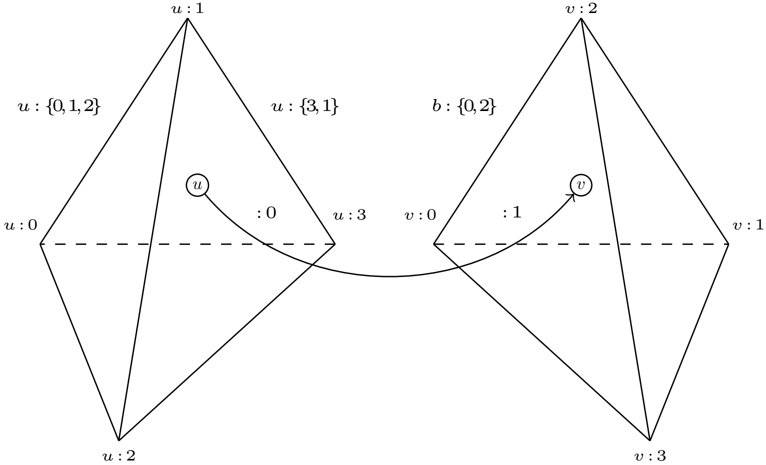

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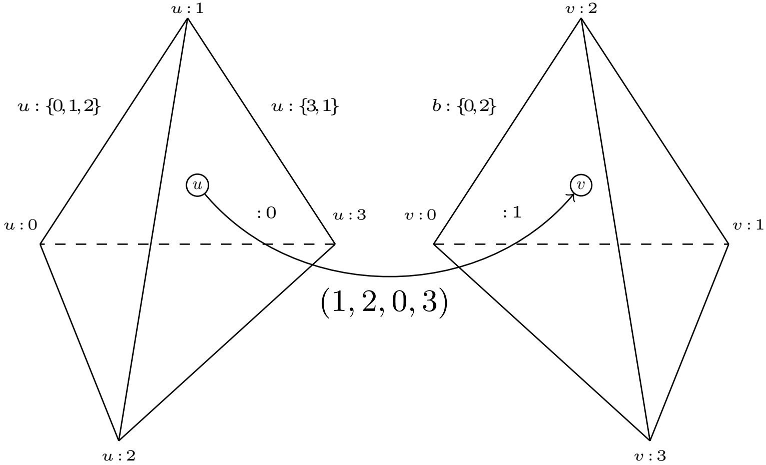

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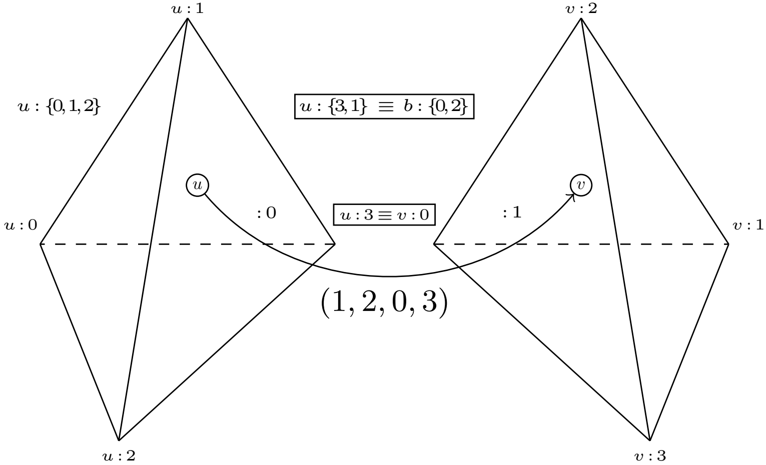
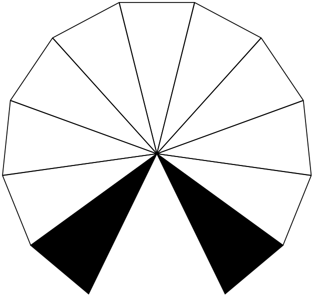
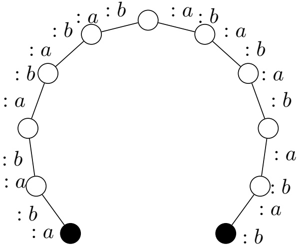


Hinging and alternating paths





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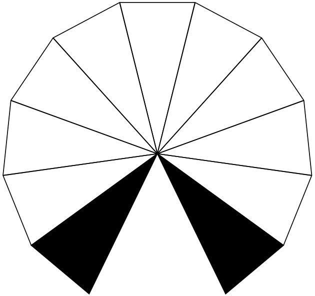
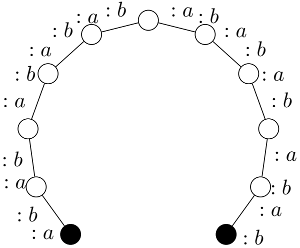


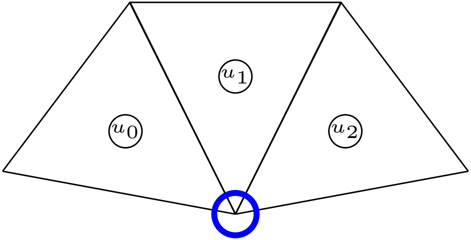


Problems:

- distance between two triangles?
- bounded density of information?
- and later: twists? manifold? pseudo-manifold?

Hinging and alternating paths

Characterizing 0-simplices:



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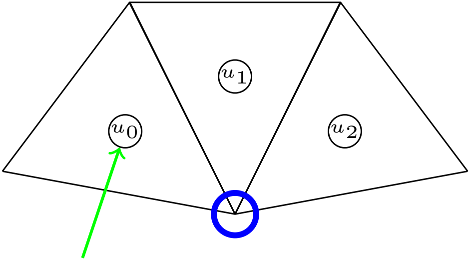

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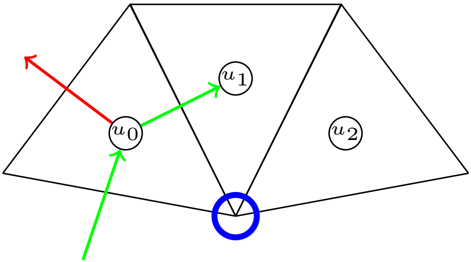

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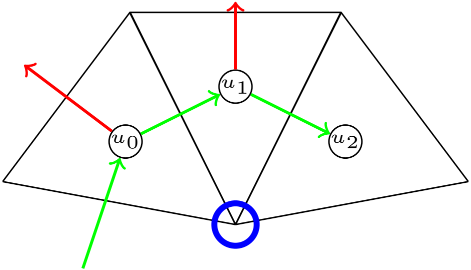

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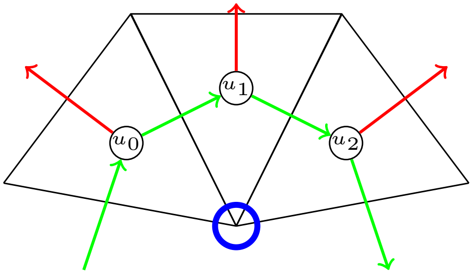
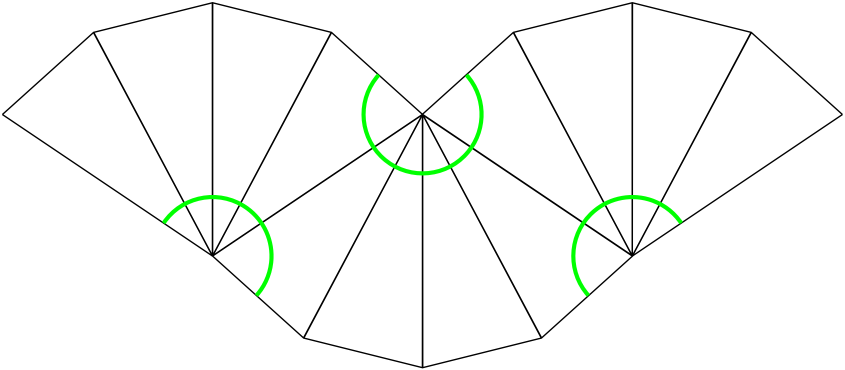
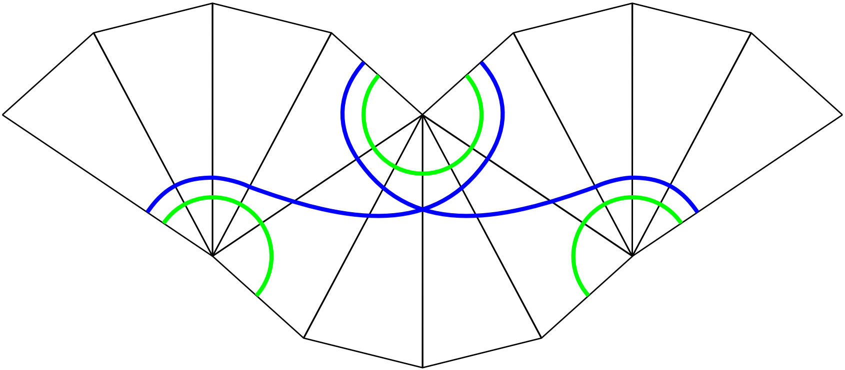


Geometric distance and alternating paths



0-alternating path $\leftrightarrow$ distance 1

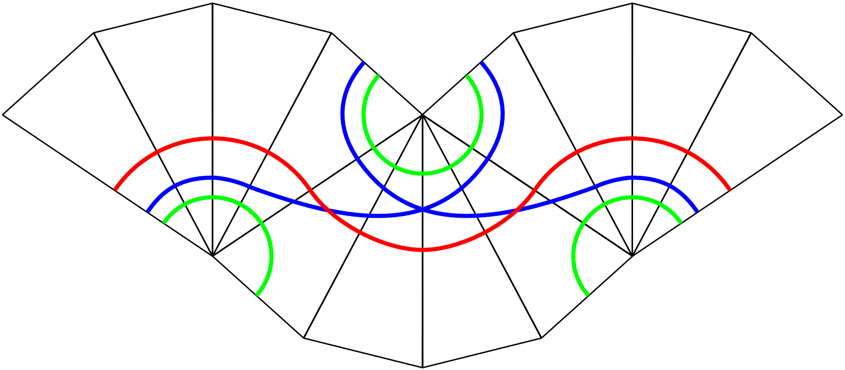
Geometric distance and alternating paths



0-alternating path $\leftrightarrow$ distance 1

1-alternating path $\leftrightarrow$ distance 2

Geometric distance and alternating paths

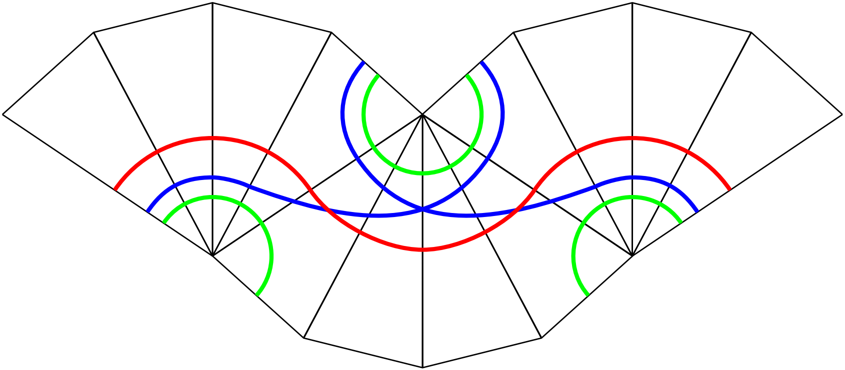


0-alternating path ↔ distance 1

1-alternating path ↔ distance 2

2-alternating path ↔ distance 3

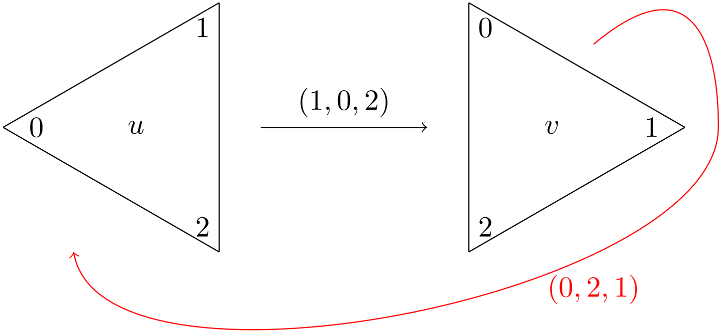
Geometric distance and alternating paths



Bounded neighbourhood $\leftrightarrow$ Bounded 0-alternating paths

Twists

How to detect edge bendings?



Just look at all hinging cycles!

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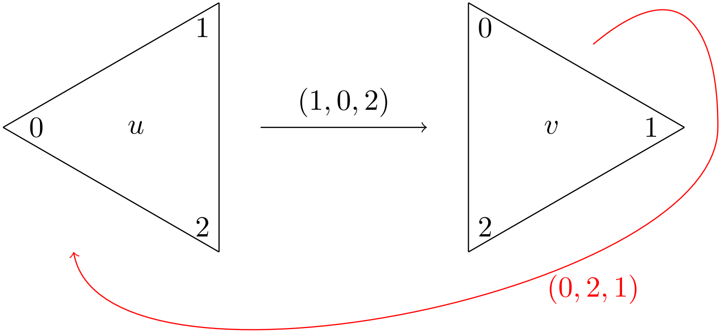


$u\!:\!1 \equiv v\!:\!0$ and $v\!:\!0 \equiv u\!:\!0 \Rightarrow u\!:\!1 \equiv u\!:\!0$

Just look at all hinging cycles!

What do we have:

- Notion of oriented complex
- Notion of bounded Neighbourhood (bounded star)

What do we need:

- Can we compare two graphs? Can we define homeomorphism?
- Is our graph a manifold?

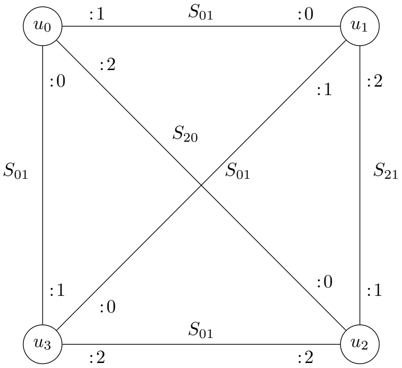
Outline

Bistellar move - Sphere

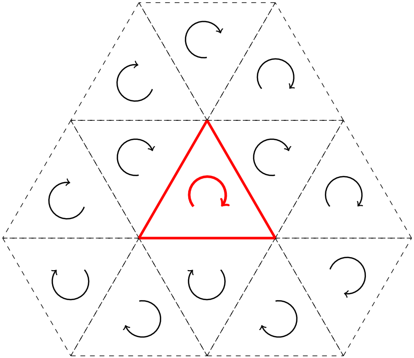
Idea in 2D: Remove one or two triangles and replace them with their complementary in a tetrahedron.

n-sphere:

- $n + 2$ *n*-simplices forming a clique
- No twists



Bistellar move



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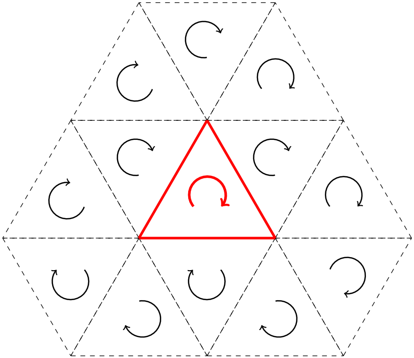

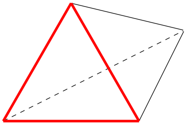

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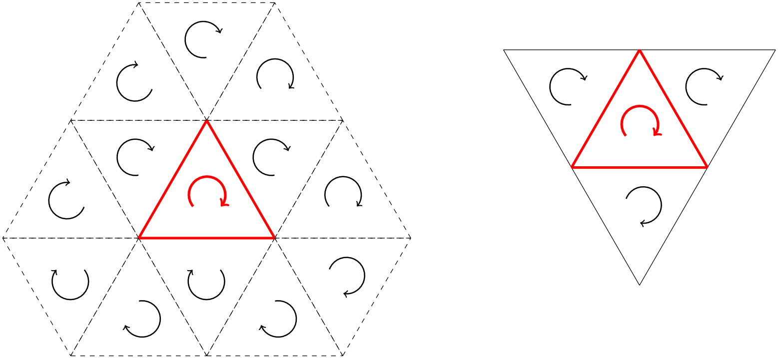

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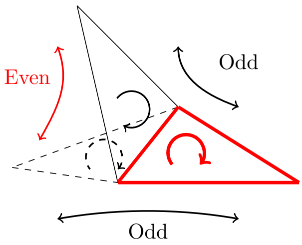

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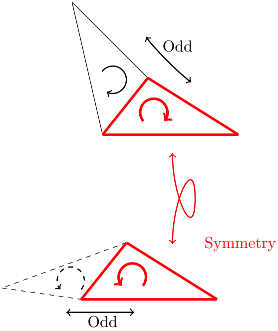

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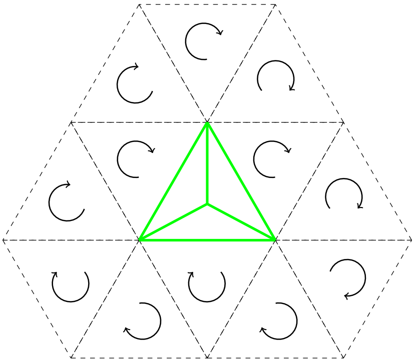
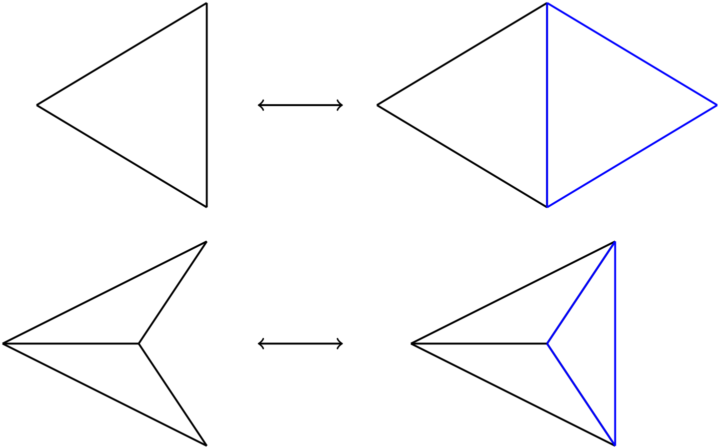


Elementary Shellings

Idea: Extend (or reduce) the border of the complex by adding a new n-simplex.

What is allowed?
Everything but filling holes and adding twists.



Pachner moves - homemorphism (Current work)

Pachner moves = Bistellar moves + Elementary Shellings + Rotations

Conjecture

Pachner moves corresponds exactly to homeomorphisms.

This allows us to:

- Look at the neighbourhood of each 0-simplex (its star).
- Decide if its a ball of dimension n.

If the previous conjecture holds, we have:

Conjecture

Given a finite graph X it is decidable to know if its interpretation as a complex is a manifold.

In particular we can define:

Definition (Manifold preserving)

A function $F : \mathcal{X}_{\{0,\dots,n\}} \to \mathcal{X}_{\{0,\dots,n\}}$ said to be manifold preserving, if

$$X \text{ manifold} \Rightarrow F(X) \text{ manifold}$$

Causal Dynamics of Discrete Manifolds

$F : \mathcal{X}_{\{0,\dots,n\}} \to \mathcal{X}_{\{0,\dots,n\}}$ causal dynamics :

- Continuous
- Translation invariant

Causal Dynamics of Discrete Manifolds

$F : \mathcal{X}_{\{0,\dots,n\}} \to \mathcal{X}_{\{0,\dots,n\}}$ causal dynamics of Discrete Manifolds:

- Continuous
- Translation invariant
- Vertex rotation commuting
- Bounded star
- Manifold preserving (Current work)

References I

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