Topological Automata

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Abstract. We give here a new, topological, definition of automata that extends previous definitions of probabilistic and quantum automata. We then prove in an unified framework that deterministic or non-deterministic probabilistic and quantum automata recognise only regular languages with an isolated threshold.
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Introduction

A result in probabilistic automata theory states that any language accepted with an isolated threshold λ (no word is accepted with a probability p in some neighbourhood of λ) is a regular language [1]. The same fact also surprisingly holds for quantum automata [2,3].

In fact, the two proofs are very similar, and it is tempting to generalise them to many others models of automata. Bozapalidis [4] gives such a generalisation, considering automata as vector spaces on which the monoid $\Sigma^\star$ acts linearly. This work allows him to give a unified proof of the fact that the two previous models of automata recognise only regular languages when limited to isolated threshold. However, no result about non-deterministic automata has been obtained in this way.

We will provide here a new way to define automata, based on some topological considerations, which will extend naturally quantum and probabilistic automata. We will then prove how some properties about topological spaces (mainly compactness and spaces with finitely many components) ensure the regularness of the languages accepted by separation, the topological equivalent on an isolated threshold.

The beauty of topological spaces is that the notion of non-deterministic automata follows naturally, and the determinisation process of classical finite automata is easy to transpose: instead of non-empty sets, which have no properties for topological spaces, we will use non-empty compact sets. We will then see that the fundamental property is also true for non-deterministic quantum or probabilistic automata: they accept only regular languages when limited to isolated threshold. This result is new for quantum automata, and provide an easier proof than [5] for probabilistic automata.

1 Topological Automata

We will assume that the reader is familiar with the classical theory of formal languages and finite automata, in particular the Myhill-Nerode theorem. We refer the reader to [6] for more details. The empty word will be denoted by ϵ.

Many models of automata behave in the same way: from an initial configuration, the automata flows through configurations by reading letters. When the last letter is read, we decide if the word is accepted by observing the configuration. We translate here this procedure in a topological fashion:

1.1 Definitions

Definition 1. *A class of automata is a tuple $(\mathcal{O}, \mathcal{C}, \mathcal{M})$ where*

- *$\mathcal{O}$ is a topological monoid : $\mathcal{O}$ is a metric space and also a monoid for which the multiplication $\circ$ is a continuous map. $\mathcal{O}$ is called the set of* operators*.*
- *$\mathcal{C}$ is a metric space; $\mathcal{O}$ right acts continuously on $\mathcal{C}$. $\mathcal{C}$ is called the set of* configurations*. The action will be denoted $\centerdot$.*
- *$\mathcal{M}$ is a set of continuous maps from $\mathcal{C}$ to $\mathbb{R}$. $\mathcal{M}$ is called the set of* measures*. Furthermore, for each $m \in \mathcal{M}$, and each operator $o \in \mathcal{O}$, the map $c \mapsto m(c \centerdot o)$ is in $\mathcal{M}$.*

We will now see how the classical definitions of automata fit in this framework

- The class of (deterministic) finite automata with n states is defined by :
  - The set of configurations is $\mathcal{C} = \{1 \ldots n\}$ also written Q.
  - The set of operators $\mathcal{O}$ is the set of maps from Q to Q. $\mathcal{O}$ is a monoid for the multiplication $(f \circ g) : c \mapsto g\left(f\left(c\right)\right)$.
    $\mathcal{O}$ right acts canonically on $\mathcal{C}$: $\centerdot : (c, f) \mapsto f(c)$
  - The set of measures $\mathcal{M}$ is the set of characteristic functions of (non-empty) subsets of Q.

  All the sets are used with the discrete topology. As all the metric spaces defined are finite, all maps are obviously continuous.
- The class of probabilistic automata with n states is defined by
  - The set of configurations is the set of row $n \times 1$ real vectors $\mathbf{v}^T$ with positive coefficients and such that $\sum \mathbf{v}_i = 1$
  - The set of operators is the set of $n \times n$ stochastic matrices. $\mathcal{O}$ right acts canonically on $\mathcal{C}$: $\centerdot : (\mathbf{v}^T, \mathcal{M}) \mapsto \mathbf{v}^T \mathcal{M}$
  - The set of measures $\mathcal{M}$ is the set of maps $\mathbf{v}^T \mapsto \mathbf{v}^T \mathbf{w}$ for all column vectors $\mathbf{w}$ with coefficients in $[0, 1]$.

  All involved spaces are naturally used with the underlying topology from $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$, so that all previously defined maps are continuous.
- We define exactly in the same manner the class of (Measure-Once) quantum automata [2] with n states.

The definition of an automaton is now natural :

Definition 2. *A deterministic automaton of a class* $(\mathcal{O}, \mathcal{C}, \mathcal{M})$ *over an alphabet* Σ *is given by :*

- *A configuration* c_0 *of* $\mathcal{C}$ *called the* initial configuration
- *An operator* $X_i \in \mathcal{C}$ *for each* $i \in \Sigma$
- *A measure* $m \in \mathcal{M}$

Definition 3. *Given an automaton* $\mathcal{A} = (c_0, (X_i)_i, m)$*, the value of a word* ω *is* $Val_{\mathcal{A}}(\omega) = m\,(c_0 \centerdot X_\omega)$*, where* X_ω *denotes* $X_{\omega_1} \circ \cdots \circ X_{\omega_{|\omega|}}$ *(*X_ϵ *is the unit of* $\mathcal{O}$*)*

These definitions are coherent in the sense that the value of a (probabilistic/quantum) finite automaton is precisely its probability acceptance.

An automaton of a class does in fact not depend on all the class : The set of configurations $\mathcal{A}$ may visit is the set $\{c_0 X_\omega, \omega \in \Sigma^\star\}$ which may be not all the configurations. It may be the case even for some finite automaton where some states cannot be reached. We now define the purged automaton :

Definition 4. *Given an automaton* $\mathcal{A} = (c_0, (X_i)_i, m)$ *of a class* $(\mathcal{O}, \mathcal{C}, \mathcal{M})$*, the domain of* $\mathcal{A}$*, denoted by* $\mathrm{Dom}(\mathcal{A})$ *is the class defined by*

- *The set of operators* $\hat{\mathcal{O}}$ *is the closure of the set* $\{X_\omega, \omega \in \Sigma^\star\}$ *with the induced topology from* $\mathcal{O}$
- *The set of configurations* $\hat{\mathcal{C}}$ *is the closure of* $c_0 \centerdot \hat{\mathcal{O}}$ *with the induced topology from* $\mathcal{C}$*.*
- *The set of measures* $\hat{\mathcal{M}}$ *is given by the maps* $c \mapsto m(c \centerdot o)$ *for* $o \in \hat{\mathcal{O}}$

The purge of $\mathcal{A}$ *is the automaton* $\hat{\mathcal{A}} = (c_0, (X_i)_i, m')$ *of the class* $\mathrm{Dom}(\mathcal{A})$ *where* m' *is the restriction of* m *to* $\hat{\mathcal{C}}$*.*

The class $\mathrm{Dom}(\mathcal{A})$ is important as every property about the automata $\mathcal{A}$ will depend on $\mathrm{Dom}(\mathcal{A})$ rather than on its class. Note that we use in the definition the closure of the sets rather than the sets themselves. This is done so that the various topological spaces conserve some properties : If $\mathcal{C}$ is compact (resp. complete), this is also the case of $\hat{\mathcal{C}}$. Note also that for a purged automaton, the set $\{X_u, u \in \Sigma^\star\}$ is by construction dense in $\hat{\mathcal{O}}$. This fact will be useful later.

1.2 Recognised languages

We may now define numerous definitions for recognising languages , based on the usual definitions for probabilistic and quantum automata :

Definition 5. *Let* $\mathcal{L}$ *be a language and* $\mathcal{A}$ *be an automaton.*

- $\mathcal{L}$ *is said to be recognised by* $\mathcal{A}$ *with threshold* λ *if* $\mathcal{L}$ *is the set of words accepted with a value greater than* λ *:* $\omega \in \mathcal{L} \iff Val_{\mathcal{A}}(\omega) > \lambda$*.*
- $\mathcal{L}$ *is said to be recognised by separation by* $\mathcal{A}$ *if there exists two disjoints closed sets* A *and* $A^\top$ *such that*
  - $\omega \in \mathcal{L} \implies Val_{\mathcal{A}}(\omega) \in A$*,*

- $\omega \notin \mathcal{L} \implies Val_{\mathcal{A}}(\omega) \in A^{\top}$,
- *If the sharper condition* $\forall x \in A \forall y \in A^{\top}, |x - y| > \epsilon$ *is verified, then* $\mathcal{L}$ *is said to be recognised by* $\mathcal{A}$ *with (bounded) error* ϵ.

The very definition of a class of automata permits us to state some closure properties for these languages

Lemma 1. *Let* $\mathcal{A}$ *be an automaton of a class* $\mathfrak{C}$. *Then*

- *For all words* $u \in \Sigma^{\star}$, *there exists an automaton* $\mathcal{B}$ *of the class* $\mathfrak{C}$ *such that* $Val_{\mathcal{B}}(\omega) = Val_{\mathcal{A}}(u\omega)$
- *For all words* $v \in \Sigma^{\star}$, *there exists an automaton* $\mathcal{B}$ *of the class* $\mathfrak{C}$ *such that* $Val_{\mathcal{B}}(\omega) = Val_{\mathcal{A}}(\omega v)$
- *For every alphabet* Δ *and every morphism* h *from* $\Delta^{\star}$ *to* $\Sigma^{\star}$, *there exists an automaton* $\mathcal{B}$ *of* $\mathfrak{C}$ *such that* $Val_{\mathcal{B}}(\omega) = Val_{\mathcal{A}}(h(\omega))$

Proof. Write $\mathcal{A} = (c_0, (X_i)_{i \in \Sigma}, m)$ and consider the automata

- $\mathcal{B} = (c_0 \centerdot X_u, (X_i)_{i \in \Sigma}, m)$,
- $\mathcal{B} = (c_0, (X_i)_{i \in \Sigma}, m')$ with $m'(x) = m(x \centerdot X_v)$
- $\mathcal{B} = (c_0, (Y_i)_{i \in \Delta}, m)$ with $Y_i = X_{h(i)}$.

□

Corollary 1 *The class of languages recognized (with one of the previous definitions) by a class of automata is closed by quotients and inverse morphism.*

Furthermore, it is trivial to see that the class of languages recognized by separation or separation with error ϵ is closed by complementation.

1.3 Regular languages

We will now give sufficient conditions for a language accepted by a topological automaton to be regular. The separating condition will be in this context precious. We have indeed the first theorem :

Theorem 2. *Let* $\mathcal{L}$ *be a language recognised by separation by an automaton* $\mathcal{A} = (c_0, (X_i)_i, m)$. *If* $\overline{\{X_i, i \in \Sigma^{\star}\}}$ *has finitely many connected components then* $\mathcal{L}$ *is regular.*

Proof. We can always change $\mathcal{A}$ into $\hat{\mathcal{A}}$. In this context,

- $\{X_i, i \in \Sigma^{\star}\}$ is dense in $\mathcal{O}$, so that for all $o \in \mathcal{O}, m(c_0 \centerdot o) \in A \cup A^{\top}$.
- $\mathcal{O}$ has finitely many connected components.

We will show that L has finitely many left quotients $u^{-1}L = \{v | uv \in L\}$.

For every $X \in \mathcal{O}$, set $M(X) = \{o | m(c_0 \centerdot X \centerdot o) \in A\}$. It is easy to see that if $M(X_u) = M(X_w)$ then $u^{-1}L = w^{-1}L$ (this assertion is in fact always true). It is then sufficient to prove that $M(X)$ can take only finitely many values.

Now, $M(X)$ is closed as the preimage of a closed set by a continuous map. However, $M(X)$ is also open as its complementary is by separation $\{o | m(c_0 \centerdot X \centerdot o) \notin A\} = \{o | m(c_0 \centerdot X \centerdot o) \in A^{\top}$ $M(X)$ is then a clopen set, hence a union of connected components. As there are finitely many components, this concludes the proof. □

As any compact group has finitely many components, this gives an alternative proof of the fact that Measure-Once quantum automata recognized only regular languages with an isolated threshold.

However, the following theorem will be in practice more useful

Theorem 3. *Let $\mathcal{L}$ be a language recognised by separation by an automaton $\mathcal{A} = (c_0, (X_i)_i, m)$. If $\overline{\{X_i, i \in \Sigma^\star\}}$ is compact (that is the monoid generated by the X_i is relatively compact), then $\mathcal{L}$ is regular.*

Proof. For compact sets, separation is the same as separation with some error ϵ, we will use this alternate formulation.

We might use the same ideas, but we will here rather proceed differently to capture precisely the role of ϵ. This will give bounds in the next theorem.

We will use the Myhill-Nerode theorem[6] and show that the relation

$$uRv \iff (\forall t, ut \in \mathcal{L} \iff vt \in \mathcal{L})$$

has finitely many classes.

We again change $\mathcal{A}$ into $\hat{\mathcal{A}}$. The hypothesis then implies that $\mathcal{C}$ and $\mathcal{O}$ are compact.

Now, as $\mathcal{O}$ is compact, the family of functions from $\mathcal{C}$ to $\mathbb{R}$ which can be given by $\{f_x : c \mapsto m(c \centerdot x), x \in \mathcal{O}\}$ is equicontinuous, and even uniformly equicontinuous, as $\mathcal{C}$ is compact : For every $\epsilon > 0$ there exists δ such that $\forall c, c' \in \mathcal{C}, \forall x \in \mathcal{O}, d(c, c') \leq \delta \implies |m(c \centerdot x) - m(c' \centerdot x)| \leq \epsilon$.

We use this assertion with the ϵ given by the fact that $\mathcal{L}$ is recognised in an isolated way.

Now, let u et v be two words such that $u \not R\, v$. There exists t such that $ut \in \mathcal{L}$, and $vt \notin \mathcal{L}$, or conversely. Then $|m(c_0 \centerdot X_{ut}) - m(c_0 \centerdot X_{vt})| > \epsilon$, that is $|m((c_0 \centerdot X_u) \centerdot X_t) - m((c_0 \centerdot X_v) \centerdot X_t)| > \epsilon$. So that we have $d(c_0 \centerdot X_u, c_0 \centerdot X_v) > \delta$.

We now just have obtained the following result : If u et v are not in the same class, then $d(c_0 \centerdot X_u, c_0 \centerdot X_v) > \delta$.

Now, by compactness, we cover $\mathcal{C}$ with finitely many balls of radius $\delta/2$. Then if u et v are such that $c_0 \centerdot X_u$ and $c_0 \centerdot X_v$ are in the same ball, then they must be in the same class. Therefore there is finitely many classes. □

A closer look at the proof shows that we can have a somehow stronger result :

Theorem 4. *Let $(\mathcal{O}, \mathcal{C}, \mathcal{M})$ be a class of automata. If $\mathcal{O}$ and $\mathcal{C}$ are compact and $\mathcal{M}$ is equicontinuous, then for every ϵ there exists a p such that any language accepted by separation with error ϵ by an automaton of this class is regular and its minimal automaton has less than p states.*

Indeed, we now have that the family of functions from $\mathcal{C}$ to $\mathbb{R}$ given by $\{c \mapsto m(c \centerdot x) : x \in \mathcal{C}, m \in \mathcal{M}\}$ is uniformly equicontinuous; given δ, we can thus set p to be the cardinal of a minimal covering of $\mathcal{C}$ by balls of radius $\delta/2$.

This stronger property is verified by classical, probabilistic and quantum automata with n states. We can furthermore show that we can take $p(n, \epsilon) = n$

for classical automata, $p(n, \epsilon) = \lceil (2/\epsilon)^n \rceil$ for probabilistic automata. A similar bound has been obtained for quantum automata [7].

We will give here an example to explain why the hypotheses are necessary.

Take the class of automata given by $\mathcal{O} = \mathbb{R}$, where the multiplication operator $\circ$ is the usual addition over $\mathbb{R}$, $\mathcal{C} = \mathbb{R}$ where $\mathcal{O}$ acts naturally over $\mathcal{C}$ by addition, and the set of measures is the set of continuous maps from $\mathbb{R}$ to $\mathbb{R}$. Now, let $\mathcal{A}$ be the automaton defined by $X_a = 1, X_b = -1$, $c_0 = 0$, $m : x \mapsto x$. Note that $\{X_i\}^\star = \mathbb{Z}$ is not relatively compact and has infinitely many connected components.

Obviously, $Val_{\mathcal{A}}(\omega) = |\omega|_a - |\omega|_b$, where for $i \in \Sigma$, $|\omega|_i$ is the number of i in ω. $\mathcal{L} = \{\omega : |\omega|_a = |\omega|_b\}$ is therefore recognised by separation by $\mathcal{A}$ but is not regular.

2 Non-deterministic automata

We will see how non-deterministic automata fit in our framework, and how rationality of the languages recognised can be easily obtained by means of the Hausdorff metric on compact sets.

2.1 Definition

As it is the case for classical finite automata, we can now define many notions of non-determinism :

- We add so-called $\mathcal{E}$-transitions.
- Instead of having just one operator per letter, we have several ones.

We will show that adding $\mathcal{E}$-transitions still produces regular languages. This is still true for the second notion of non-determinism, and the demonstration is very similar to the proof we will give here.

Let us define precisely this notion.

Definition 6. *A non-deterministic automaton $\mathcal{A}$ over Σ of a class of automata is given by a deterministic automaton $\mathcal{B}$ over $\Sigma \cup \{\mathcal{E}\}$ where the value of a word $\omega \in \Sigma$ is now defined as :*
$Val_{\mathcal{A}}(\omega) = \sup_{h(u)=\omega} Val_{\mathcal{B}}(u)$
where h is the morphism that maps $x \in \Sigma$ to x and $\mathcal{E}$ to the empty word ϵ.

We can express it intuitively in the following way : the automaton acts as a normal automaton, except $\mathcal{E}$-transition can occur every time; The value of a word is then the maximum of all values we can obtain.

We can also define the value in the following manner :

Proposition 1. *Define recursively*

- $K_\epsilon = c_0 \cdot \{X_{\mathcal{E}}\}^\star = \left\{c_0 \cdot x : x \in \{X_{\mathcal{E}}\}^\star\right\}$
- *for* $i \in \Sigma$, $K_{vi} = K_v \cdot X_i \cdot \{X_{\mathcal{E}}\}^\star = \left\{c \cdot X_i \cdot x : x \in \{X_{\mathcal{E}}\}^\star, c \in K_v\right\}$

Then $Val_{\mathcal{A}}(\omega) = \sup m(K_\omega)$.

It can easily be shown that we can replace $\{X_{\mathcal{E}}\}^\star$ by $\overline{\{X_{\mathcal{E}}\}^\star}$. Indeed, if we call J_ω the sets obtained this way, then $K_\omega \subseteq J_\omega \subseteq \overline{K_\omega}$, so that $\sup m(K_\omega) = \sup m(J_\omega)$.

We will now prove that if $\{X_i\}^\star_{i\in\Sigma\cup\{\mathcal{E}\}}$ is relatively compact and $\mathcal{L}$ recognised by separation, then $\mathcal{L}$ is regular.

Please note that, as the language over $\Sigma \cup \{\mathcal{E}\}$ is not necessarily separated, we can not use directly the results from the previous section.

We will in fact determinise the automaton. For classical automata, this involves sets of configurations. The natural, topological equivalent will be compact sets of configurations.

2.2 Hausdorff metric

Unless specified, all compact sets will be taken non-empty.

There exists a natural topology on compact sets of a metric space, given by the Hausdorff metric :

Definition 7. *Given a metric space* (E, d)*, and two non-empty compact sets* K_1 *and* K_2 *of* E*, the Hausdorff distance is defined by*

$$d_H(K_1, K_2) = \max_{x_1,x_2\in K_1\times K_2} \max d(x_1, K_2), d(x_2, K_1)$$

where $d(x, K)$ *is the distance of* x *from* K : $d(x, K) = min_{y\in K} d(x, y)$

d_H is a metric over the set of non-empty compact sets of E, denoted by $\mathfrak{E}$. This space is called the Hausdorff metric space induced by E.

This metric extends naturally the metric d. In fact, we can "lift" the functions :

Proposition 2. *Let* (E, d) *and* (F, d') *be two metric spaces, and* $(\mathfrak{E}, d_H)$, $(\mathfrak{F}, d'_H)$ *their induced Hausdorff metric spaces.*

Let $f : E \to F$ *be a continuous map.*

Then $\mathfrak{f} : \begin{array}{l} \mathfrak{E} \to \mathfrak{F} \\ K \mapsto f(K) \end{array}$ *is continuous.*

Similar results of this kind will be used in the following section. We will only prove this fact, but the proof contains in substance all the other results.

Proof. First note that $\mathfrak{f}$ takes values in $\mathfrak{F}$, so that $\mathfrak{f}$ is well defined.

Let K be a compact of E and $\epsilon > 0$.

For every $x \in K$, there exists $\delta_x > 0$ such that if $y \in E$ and $d(x, y) \le \delta_x$, then $d'(f(x), f(y)) \le \epsilon$. Covering K with balls of centre x and radius $\delta_x/2$, we see by compactness that there exists $\delta > 0$ such that if $x \in K$ and $y \in E$ verify $d(x, y) \le \delta$, then $d'(f(x), f(y)) \le 2\epsilon$ (Please note that we have obtained $y \in E$ and not $y \in K$).

Now, let N be a compact such that $d_H(N, K) \le \delta$.

Let $x \in N$. There exists $y \in K$ such that $d(x,y) \leq \delta$, so that we obtain $d'(f(x), f(y)) \leq 2\epsilon$. Then $d'(x, f(K)) \leq 2\epsilon$.

Let $y \in K$. There exists $n \in N$ such that $d(x,y) \leq \delta$, so that we have $d'(f(x), f(y)) \leq 2\epsilon$. Then $d'(y, f(N)) \leq 2\epsilon$.

Then, $d'_H(f(K), f(N)) \leq 2\epsilon$.

We have shown that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every N such that $d_H(N,K) \leq \delta$, we have $d'_H(f(K), f(N)) \leq 2\epsilon$. $\square$

Now, the next important result we need is the following, whose proof can be found in [8]

Theorem 5. *The set of compact sets of a compact metric space is compact with respect to the Hausdorff metric.*

2.3 Determinization

We now have all we need to determinise our automata.

Given a class $\theta = (\mathcal{C}, \mathcal{O}, \mathcal{M})$ of automata, we define the class of θ-determinised automata by :

- $\mathcal{C}'$ are the compact sets of $\mathcal{C}$ with the Hausdorff metric.
- $\mathcal{O}'$ are the compact sets of $\mathcal{O}$ with the Hausdorff metric. $\mathcal{O}'$ is a monoid for the multiplication $X \circ Y = \{x \circ y, x \in X, y \in Y\}$ which is well defined. $\mathcal{O}'$ acts naturally over $\mathcal{C}'$: $C \centerdot X = \{c \centerdot x, c \in C, x \in X\}$. Using variants of proposition 2, we can show easily that the multiplication and the action are continuous maps.
- $\mathcal{M}'$ is the set of functions $C \mapsto \sup_{c \in C} m(c)$ for $m \in \mathcal{M}$. Again, we may prove that these functions are continuous by similar arguments.

Given a non-deterministic automaton $\mathcal{A} = \{c_0, \{X_i\}_{i \in \Sigma}, X_{\mathcal{E}}, m)$ of the class θ, where $K = \overline{\{X_i, i \in \{\mathcal{E}\}^\star\}}$ is compact, we define the deterministic automaton $\mathcal{B}$ of the class of θ-determinised automata by

- the initial configuration is $C_0 = c_0 \centerdot K = \{c \centerdot k, k \in K\}$
- the operators are $X'_i = X_i \circ K = \{X_i \circ k, k \in K\}$
- the measure is now $m' : C \mapsto \sup_{c \in C} m(c)$

Proposition 3. *For any word ω, $Val_{\mathcal{A}}(\omega) = Val_{\mathcal{B}}(\omega)$*

Proof. This is obvious using the previous definition of $\mathcal{B}$ and the proposition 1 $\square$

We denote $\Sigma_{\mathcal{E}}$ for $\Sigma \cup \{\mathcal{E}\}$. We can now conclude :

Theorem 6. *Let $\mathcal{L}$ be a language recognised in an isolated way by a non-deterministic automaton $\mathcal{A} = (c_0, (X_i)_i, X_{\mathcal{E}}, m)$.*

If $\overline{\{X_i, i \in \Sigma_{\mathcal{E}}^\star\}}$ is compact, then $\mathcal{L}$ is regular.

Proof. As $K = \overline{\{X_i, i \in \{\mathcal{E}\}^\star\}}$ is compact and stable by multiplication, we can define $\mathcal{B}$, the deterministic automata corresponding to $\mathcal{A}$.

Now, it suffices to show that $M = \{X'_i, i \in \Sigma^\star\}$ is relatively compact.

Let note $E = \overline{\{X_i, i \in \Sigma^\star_{\mathcal{E}}\}}$.

Using theorem 5, we see that $\mathfrak{C}$, the set of compacts of E is compact with respect to the Hausdorff metric. Every element of M is in $\mathfrak{C}$, so that M is a subset of $\mathfrak{C}$, hence $\overline{M}$ is compact. □

We have in particular proven that non-deterministic probabilistic and quantum automata accept only regular languages when limited to isolated acceptance. These results are new for quantum automata; This also gives an alternative to the proof from [5] for probabilistic automata.

Finally note that if we can cover a compact C with n balls of radius δ, it is easy to show that we can cover the set of non-empty compact sets of C with $2^n - 1$ balls of radius δ. We can then obtain a constructive proof of the statement. We have proven for deterministic automata of a certain class that the number of states is bounded by the number of balls of the set of configurations of radius δ, where δ depends on ϵ. A detailed analysis of the previous proof shows that in fact we have the same dependency over ϵ. That is : The number of states for languages recognized by non-deterministic topological automata is bounded by the number of balls of $\mathcal{C}'$ of radius δ for the same δ. This sums up in the theorem

Theorem 7. *Let $(\mathcal{O}, \mathcal{C}, \mathcal{M})$ be a class of automata. If $\mathcal{O}$ and $\mathcal{C}$ are compact and $\mathcal{M}$ is equicontinuous, then for every ϵ there exists a p such that*

- *any language accepted by separation with error ϵ by an automaton of this class is regular and its minimal automaton has less than p states.*
- *any language accepted by separation with error ϵ by a non-deterministic automaton of this class is regular and its minimal automaton has less than $2^p - 1$ states.*

Furthermore we can take $p(n, \epsilon) = n$ for classical automata, $p(n, \epsilon) = \lceil (2/\epsilon)^n \rceil$ for probabilistic automata, where n is the number of internal states.

It would be interesting to simulate non-deterministic probabilistic or quantum automata by their deterministic counterpart, and show how efficient (in term of states) this simulation is. However, languages accepted by non-deterministic quantum automata cannot always be accepted by deterministic quantum automata, so that this simulation is not always possible.

Conclusion

We have given a new definition of automata which extends the notions of classical, probabilistic or quantum automata, and have shown how some properties of the languages they accept is just a consequence of properties of the topologies, mainly compactness. This framework also gives us a way to deal with non-deterministic automata, for which we have proven that they accept only regular languages by separation provided some compactness of the set of operators. This gives an alternate, more simple, proof of the fact that non-deterministic probabilistic automata recognise only regular languages by separation.

It would be interesting to see how other topological concepts fit in this framework and how some may correspond to other classes of languages. A good candidate is locally compact spaces. However there are not that many classes of languages which have as good characterizations as regular languages : A characterization in terms of machinery is not easy to use in the context of topological spaces.

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