Topology of singular surfaces, applications to visualization and robotics

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Scientific context

The description of singular varieties (self-intersecting curves or surfaces for example) arises in a wide range of applications, from scientific visualization to the design of robotic mechanisms. Currently, most software provide either (a) numerical approximations without topological guarantee; or (b) rely on symbolic computer algebra methods to handle singular varieties but suffer from efficiency in practice. Computing the topology of a variety consists in computing a piecewise-linear graph or triangulation that can be deformed continuously toward the input variety. In particular, it should preserve the number of connected components, the self-intersection points, etc. This subject addresses the design and the implementation of algorithms based on interval arithmetic to compute the topology of restricted classes of singular varieties.

In the case of a smooth variety, several numerical methods can guarantee its topology. One can mention global subdivision [5, 9], or continuation approaches [6]. For a singular variety, computing its topology requires i) to detect its singularities, ii) to compute the topology in a neighborhood of those singularities, iii) to compute the topology of the smooth remaining part of the variety. State-of-the-art methods to compute it are based on symbolical computer algebra algorithms such as the Cylindrical Algebraic Decomposition (see [8] and references therein). In the general case, no numerical method can yet handle the detection of the singularities, neither the computation of their local topology.

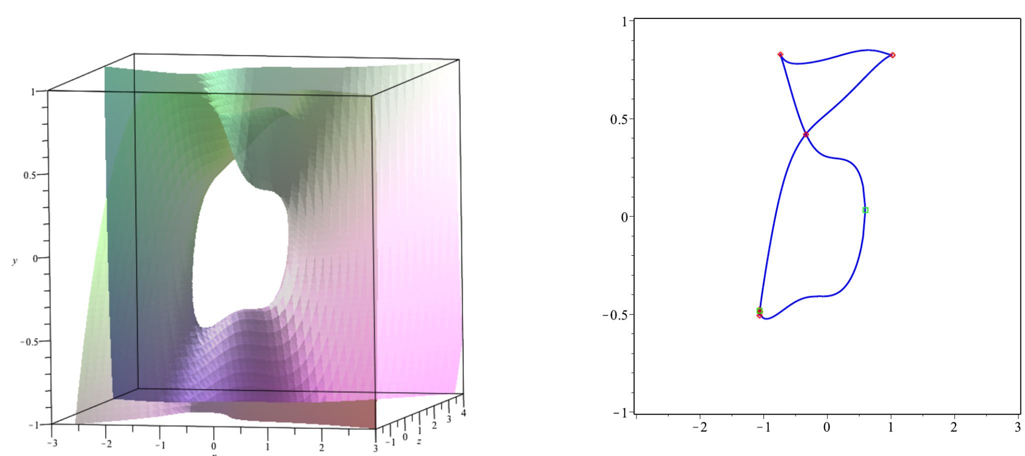

Figure 1. Left: a surface $f(x, y, z) = 0$. Its silhouette curve is defined by the system $f = \frac{\partial f}{\partial z} = 0$. Right: the projection of the silhouette is singular with node and cusp singularities.

Motivation

The singular curves and surfaces appearing in several applications often belongs to a restricted class of singular varieties. For instance, when visualizing a surface, it is natural to compute its apparent contour. Although it might be a singular curve, it is often the projection of a smooth curve, called the silhouette or polar-variety of the surface (Figure 1). In this case, we are interested in describing singular curves in the plane that are projection of smooth curves.

More generally, a natural class of singular surfaces is the apparent contour of a smooth variety of dimension 3 of $\mathbb{R}^n$. These surfaces appear naturally when visualizing surfaces and modelling robotic mechanisms with three degrees of freedom. From a mathematical point of view, their singularities have been studied and classified [1, 2]. Yet, no numerical method handles the detection of these singularities nor the computation of their local topology.

Missions

We consider an analytic smooth subvariety of dimension 3 in $\mathbb{R}^n$ defined by $f_1 = \cdots = f_{n-3} = 0$, and we want to compute the topology of its apparent contour $\mathcal{S} \subset \mathbb{R}^3$. Computing the topology of $\mathbb{S}$ requires to isolate its singularity locus. To isolate the singularities of $\mathcal{S}$, we want to use a numerical method based on Newton algorithm and interval arithmetic. More precisely, given a regular system in complete intersection, and a box B, the evaluation of a Newton operator on B can allow us to decide if B contains or not a solution (see [7, 10] and references therein). Unfortunately the natural system defining the singularities of the apparent contour $\mathcal{S}$ is not in complete intersection and we cannot guarantee its solutions with numerical methods based on Newton directly.

Given the results on the classification of singularities in [2], we expect that our recent work [3, 4] on the apparent contour of subvarieties of dimension 2 extends to the case of subvarieties of dimension 3. Moreover, the candidate will also address the problem of computing a triangulation isotopic to $\mathcal{S}$, while handling correctly the singular locus.

Profile of the candidate

The candidate should have a taste for both mathematics (geometry or numerical analysis) and computer science. Programming skills would be appreciated.

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