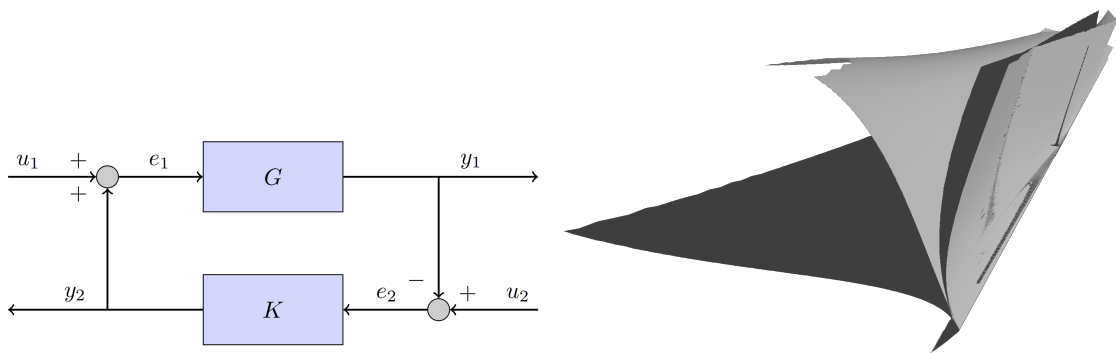


Reliable triangulation of surfaces that appear in robust control of time-delay systems

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1 Scientific context

Different problems in control theory give rise to the issue of studying the extremal solutions of algebraic or smooth/analytic curves and surfaces. Using provably correct triangulation methods, our aim is to develop new, specialised methods for studying these control problems, as well as new, reliable visualisation tools and software. Our main focus is on surfaces that are solutions of equations arising in the design of robust controllers for differential and differential-time delay systems (studied as quasipolynomials).

1.1 Robust control

A major problem in control theory is designing a controller that can stabilise an unstable system while ensuring the desired performance of the closed-loop system (i.e. the system and controller connected in a feedback loop) [13]. A key principle in control theory is that a compromise must be made between the performance and robustness of the closed-loop system (i.e. how much the controller can stabilise the system in the presence of uncertainties). Within *robust control*, H_∞ methods can be used to study this trade-off clearly, and in particular to compute the so-called *robust radius* γ of the closed-loop system [13, 6]. For ordinary differential linear systems, *reliable symbolic-numeric methods* have recently been used to initiate a *parametric robust control* [8, 9, 10], which allows the robust radius γ to be determined in terms of the system physical parameters $\theta = (\theta_1, \dots, \theta_r)$ (e.g. masses and lengths). This can be achieved by examining the eigenvalues of

a product of two positive-definite matrices, X and Y , which depend on θ (and are the solutions of two algebraic Riccati equations), i.e., by considering the following hypersurface

$$\det(\gamma^2 I - XY) = 0, \quad (1)$$

in the space formed by γ and $\theta = (\theta_1, \dots, \theta_r)$, where I denotes the identity matrix.

Focusing on two system parameters $\theta = (\theta_1, \theta_2)$, our goal is to reliably triangulate the surface S defined as the set of solutions of the following surface:

$$\det(\gamma^2 I - X(\theta_1, \theta_2)Y(\theta_1, \theta_2)) = 0. \quad (2)$$

This surface usually contains singular points [10], which makes it difficult to draw and thus prevents a clear analysis of how the robust radius varies locally in terms of θ . Furthermore, new control schemes can be developed using a reliable knowledge of the corresponding surfaces [8, 9, 10].

1.2 Differential time-delay systems viewed as quasipolynomials

A second issue in the direction of control theory is to extend the above results to the class of linear time-invariant differential constant delay systems. In the frequency domain, such systems are defined by *quasipolynomial*, i.e., analytic functions of the form

$$p(s, e^{-\tau s}) = \sum_{0 \leq i \leq n, 0 \leq j \leq m} a_{i,j} s^i e^{-\tau j s}, \quad (3)$$

where $\tau \in \mathbb{R}_{>0}$ is the delay amplitude and $s \in \mathbb{C}$ (instead of polynomials in s as in Section 1.1 for ordinary differential systems).

The quasipolynomial under consideration can correspond to the dynamics of a closed-loop system formed by a differential time-delay system and a rational/differential time-delay controller with a few unfixed parameters. In this case, the above real polynomial system depends on a few more real variables, namely, the system's and controller's parameters $\theta = (\theta_1, \dots, \theta_r)$.

The *asymptotic stability* of the closed-loop system with respect to the delay amplitude, τ , can be studied by analysing how the complex solutions of (3) vary with respect to τ , and particularly whether they can cross the imaginary axis, $i\mathbb{R}$, of $\mathbb{C}$, i.e., whether there exists $(\omega, \tau) \in \mathbb{R} \times \mathbb{R}_+$ such that $p(i\omega, e^{-i\tau\omega}) = 0$. For more details, see [2]. For one parameter (resp., two parameters), we are led to a curve in $\mathbb{R}^3$ (a surface in $\mathbb{R}^4$). In particular, the behaviour of the parameter pair (ω, τ) can be studied around some bounded regions of the parameter space using provably correct triangulation methods. This allows reliable asymptotic stability tests to be performed for linear differential time-delay systems. Furthermore, extra system's parameter θ can also be considered.

Finally, the problem described in Section 1.1 can be extended to differential time-delay systems. For more details, see [6] and the references therein.

2 Reliable triangulation of surfaces

The problem of triangulating a surface defined by an equation has been extensively studied in the literature. Notably, the book by Raphael Wenger [12], summarises several methods of triangulating

surfaces. When all the points on the surface are regular, it is also possible to use numerical algorithms to verify the correctness of the computed triangulation topology (see, for example, [11, 7]).

However, if the surface contains singular points, such as self-intersections, then most existing methods only guarantee the correctness of the resulting triangulation if the input equation is polynomial (see, for example, [1]). Unfortunately, these approaches cannot handle the exponential functions that appear in equations arising in delayed systems (see Section 1.2). Furthermore, these approaches are impractical for the high degrees that appear in robust control problems.

In some cases, guaranteeing the topology of surfaces with singular points can be achieved more efficiently. In particular, if the singular points remain singular after the parameters of the input equations are perturbed, it may be possible to use numerical algorithms to improve state-of-the-art methods for triangulating singular curves or surfaces (see, for example, [4, 3, 5]). Furthermore, these approaches could handle exponential functions that appear in differential time-delay systems.

The goals of this project are firstly to classify the stable singularities that appear in equations (2) and (3), and secondly to design an efficient algorithm to triangulate the implicit surface solution of the equations. Finally, the results obtained will be applied to propose new methods for the robust control of differential and differential-time delay systems.

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