Modèles d'environnements & planification de trajectoire

Delaunay (2 séances)

Euler's relation.

$$n \quad - \quad e \quad + \quad f \quad = 2$$

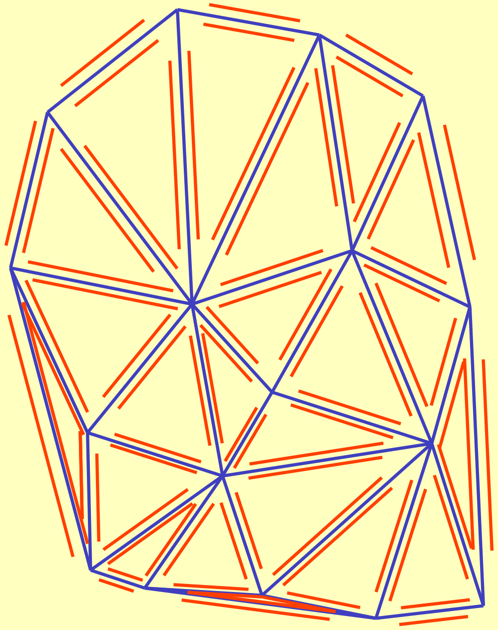
Vertices Edges Faces

$$n - e + (t+1) = 2$$

$$k + 3t = 2e$$

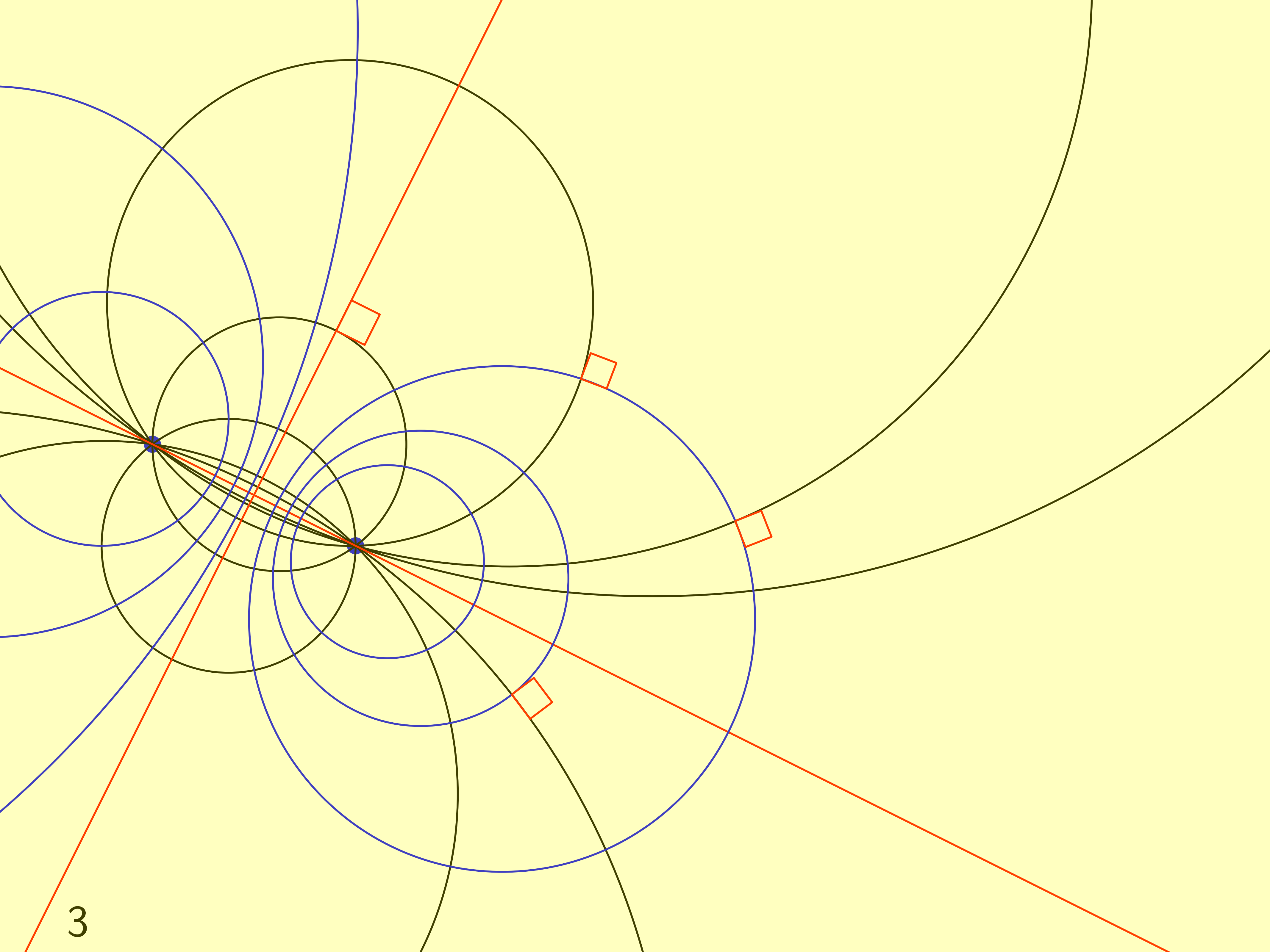
$t = \#$ triangles

$k = \#$ vertices on the convex hull

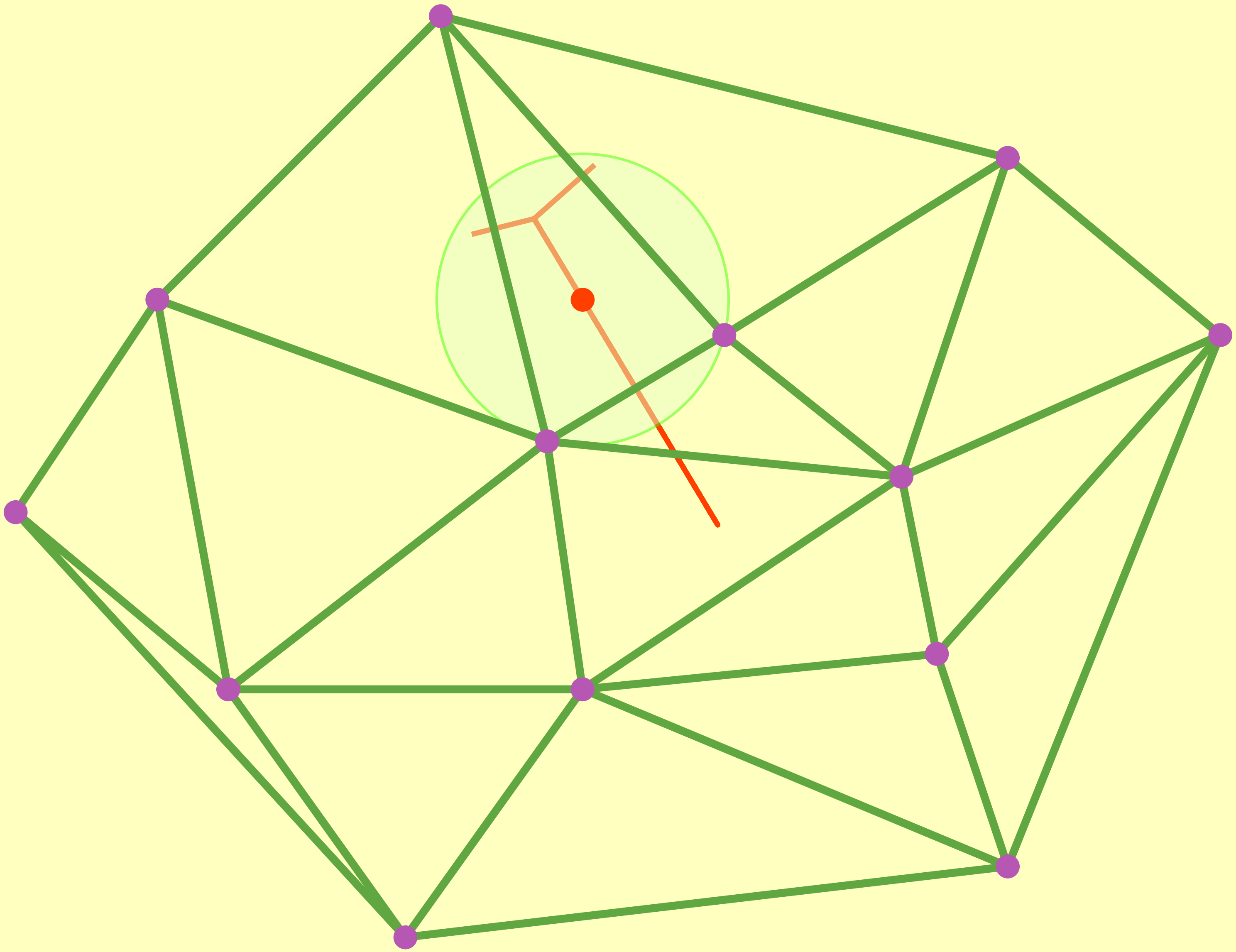


$$t = 2n - k - 2 < 2n$$

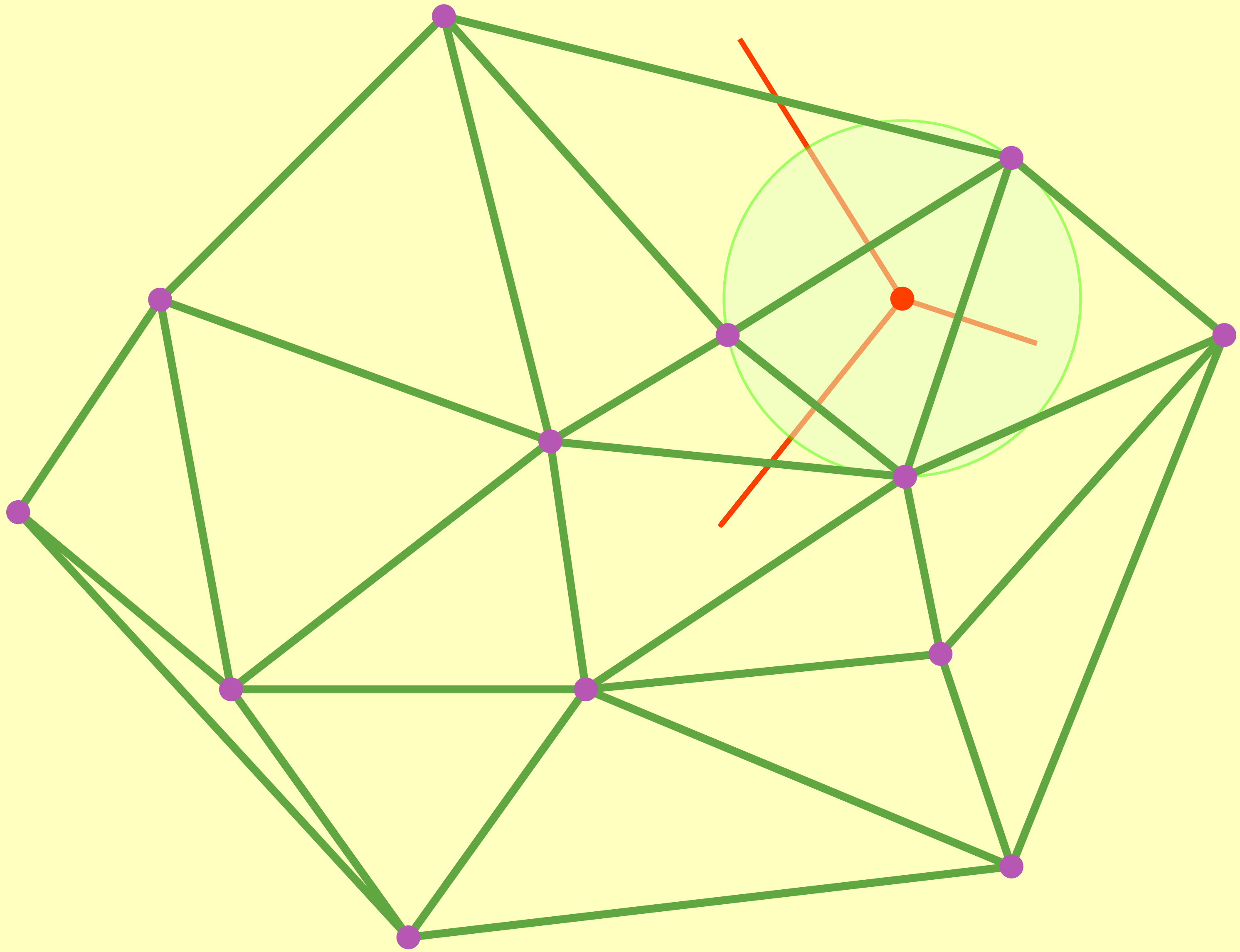
$$e = 3n - k - 3 < 3n$$



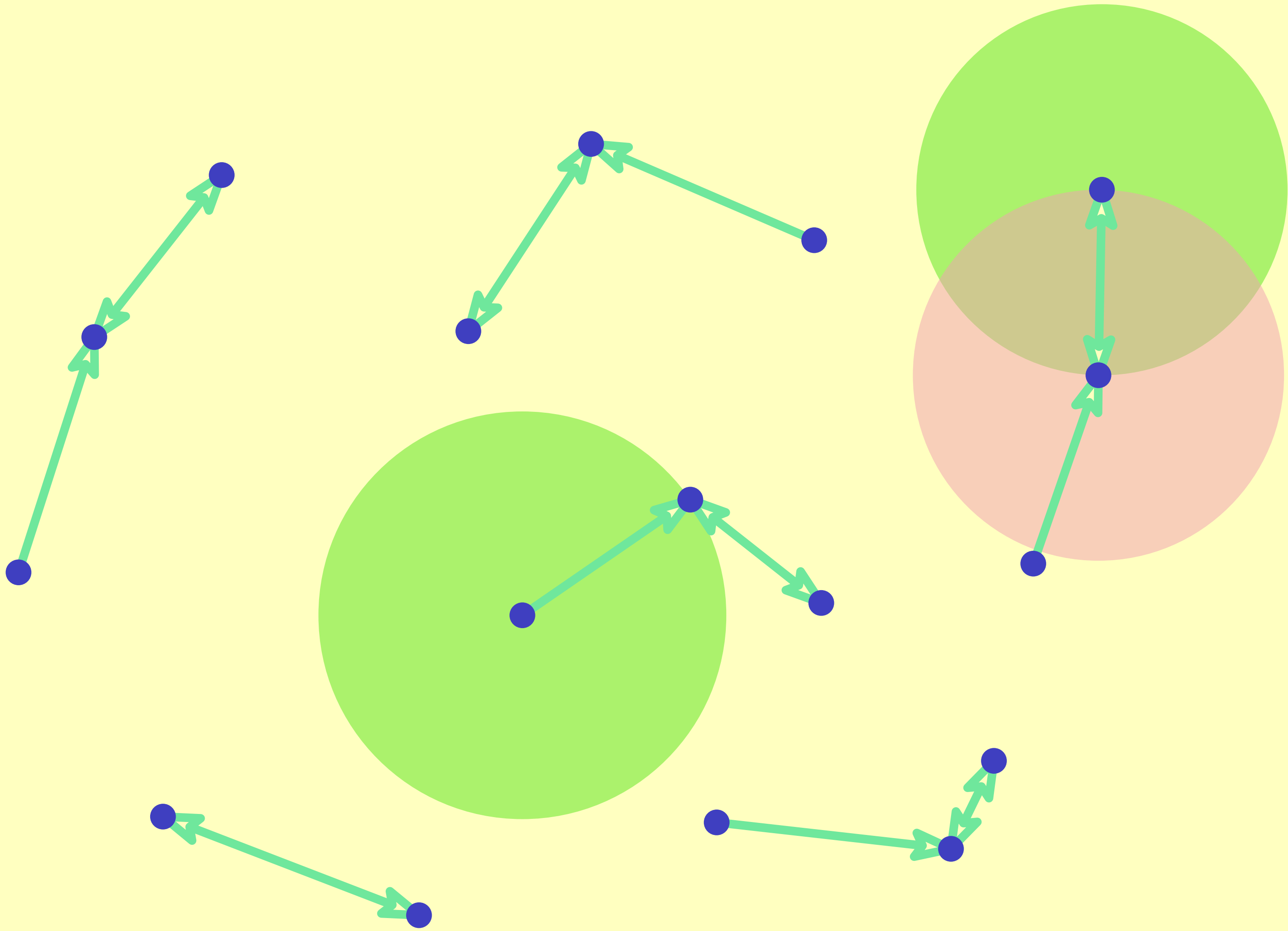
Delaunay Triangulation: definition, empty circle property



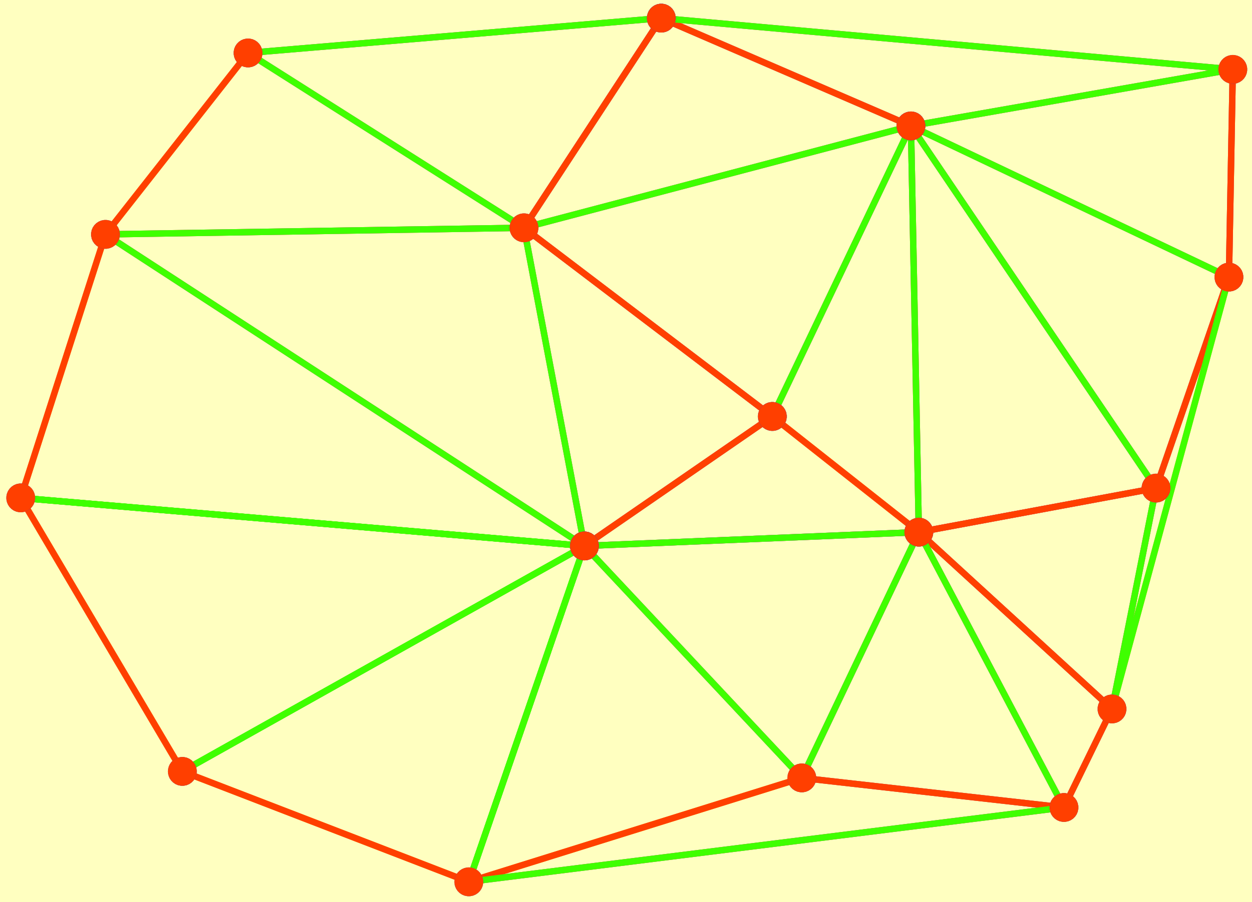
Delaunay Triangulation: definition, empty circle property



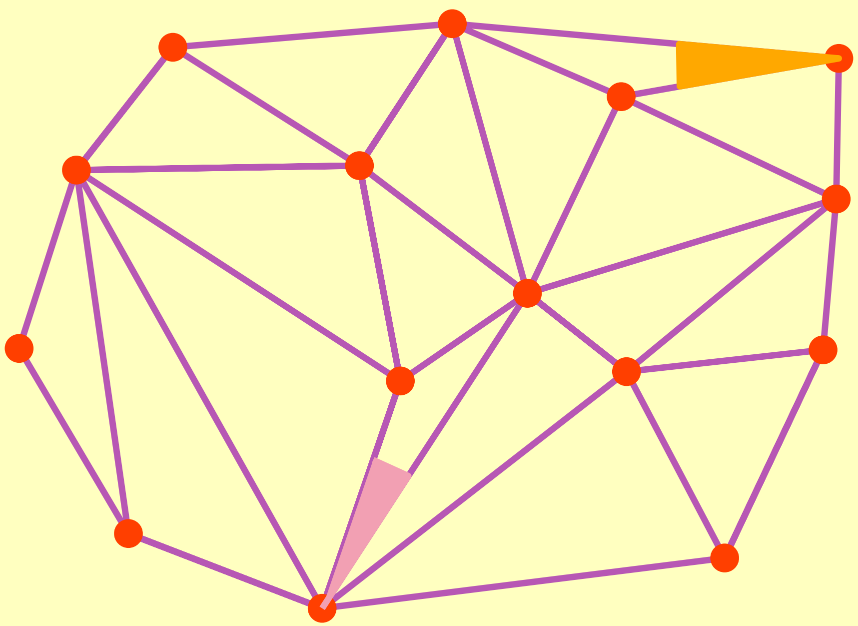
Delaunay Triangulation: Nearest Neighbor Graph



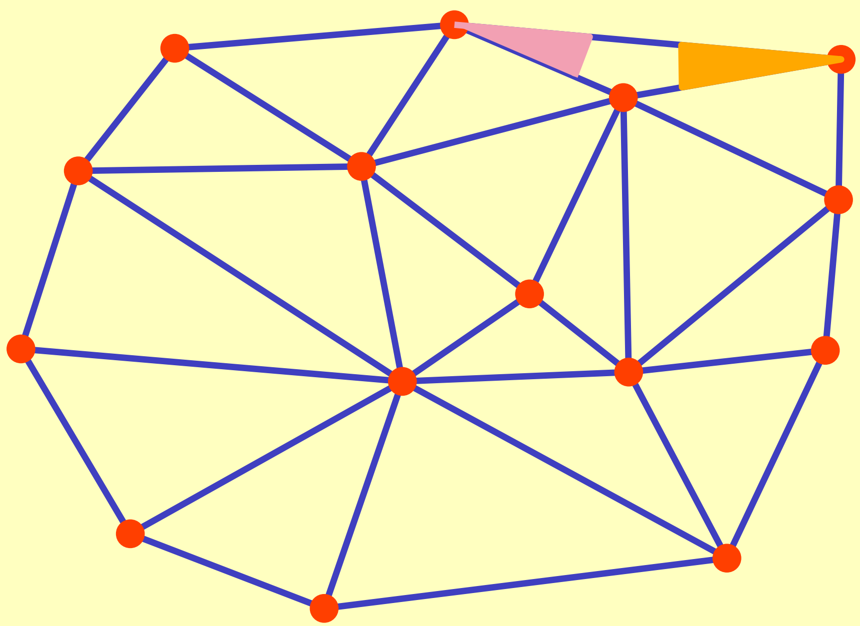
Delaunay Triangulation: EMST



Delaunay Triangulation: max-min angle



Triangulation

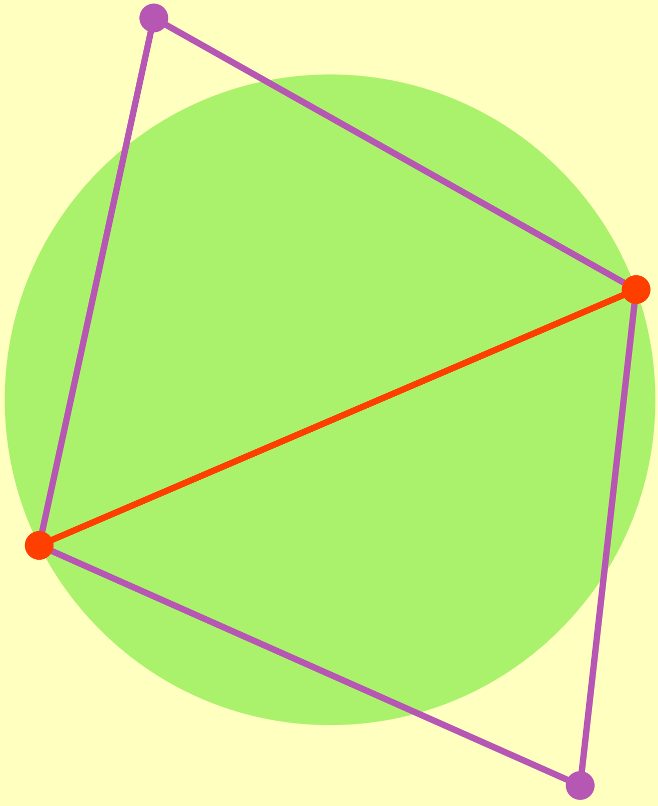


Delaunay

smallest angle

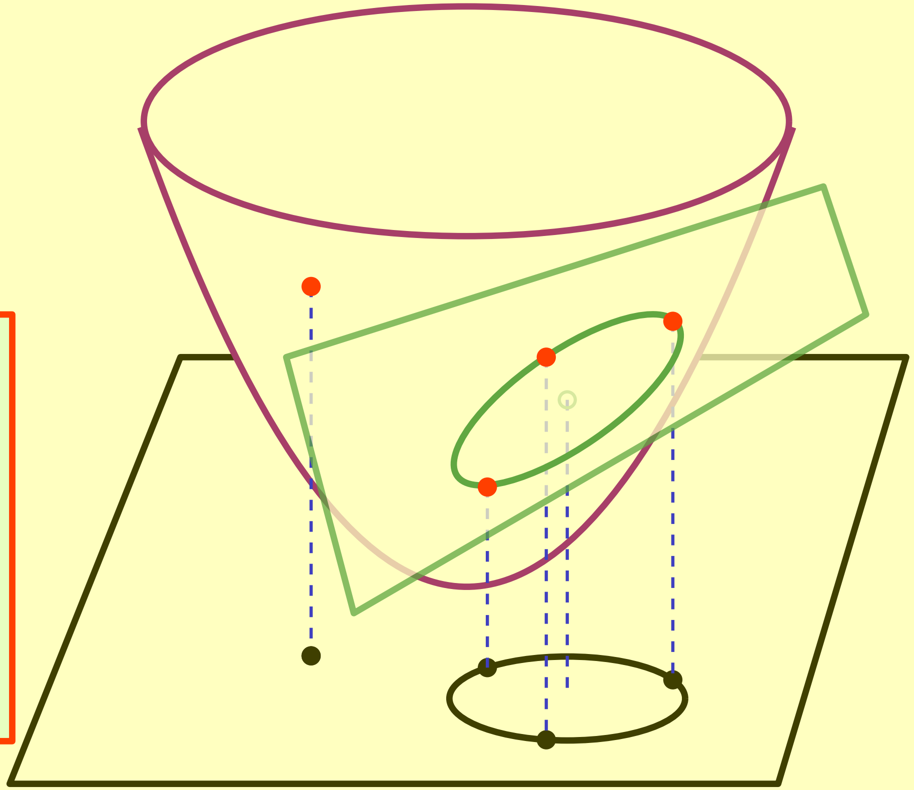
second smallest angle

Lemma $(\forall \text{ edge: locally Delaunay}) \iff \text{Delaunay}$



Delaunay Triangulation: indisk predicate

Space of circles



s inside/outside of circle through pqr

$\rightsquigarrow$ plane through $p^\star q^\star r^\star$ above/below $s^\star$

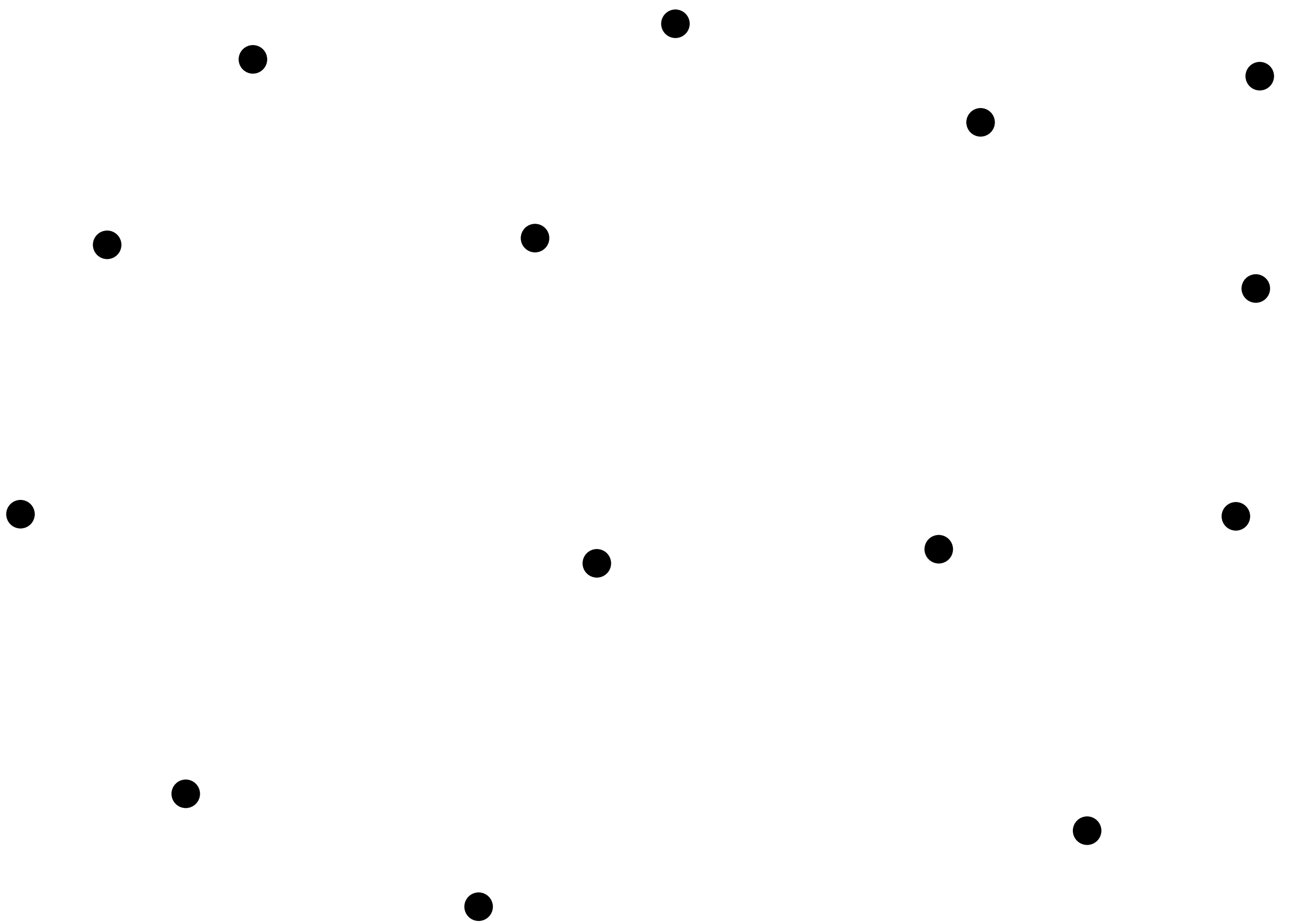
indisk predicate

$\rightsquigarrow$ 3D orientation predicate

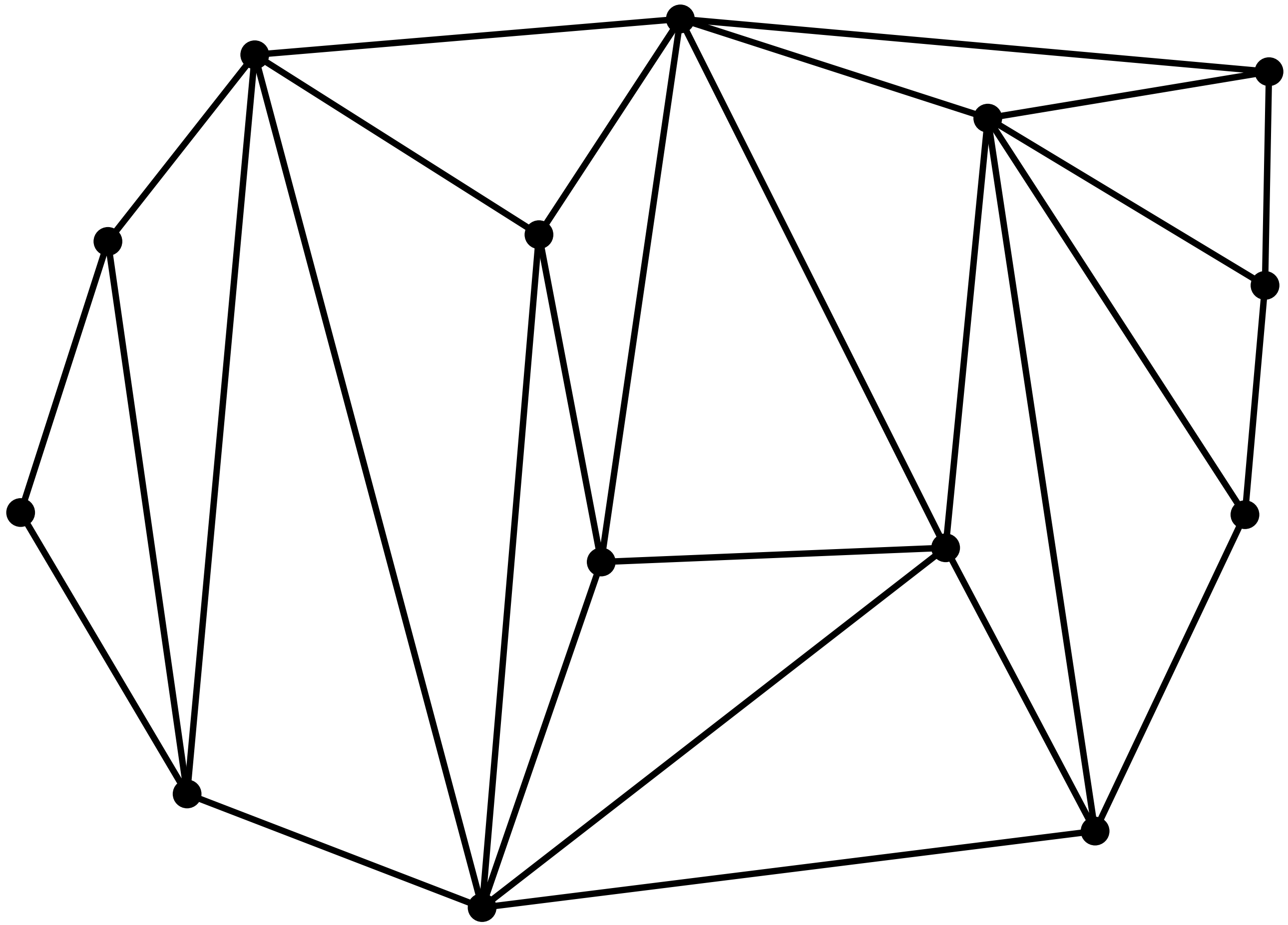
$$\text{sign}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ x_p^2+y_p^2 & x_q^2+y_q^2 & x_r^2+y_r^2 & x_s^2+y_s^2 \end{vmatrix}$$

Algorithm: flip!

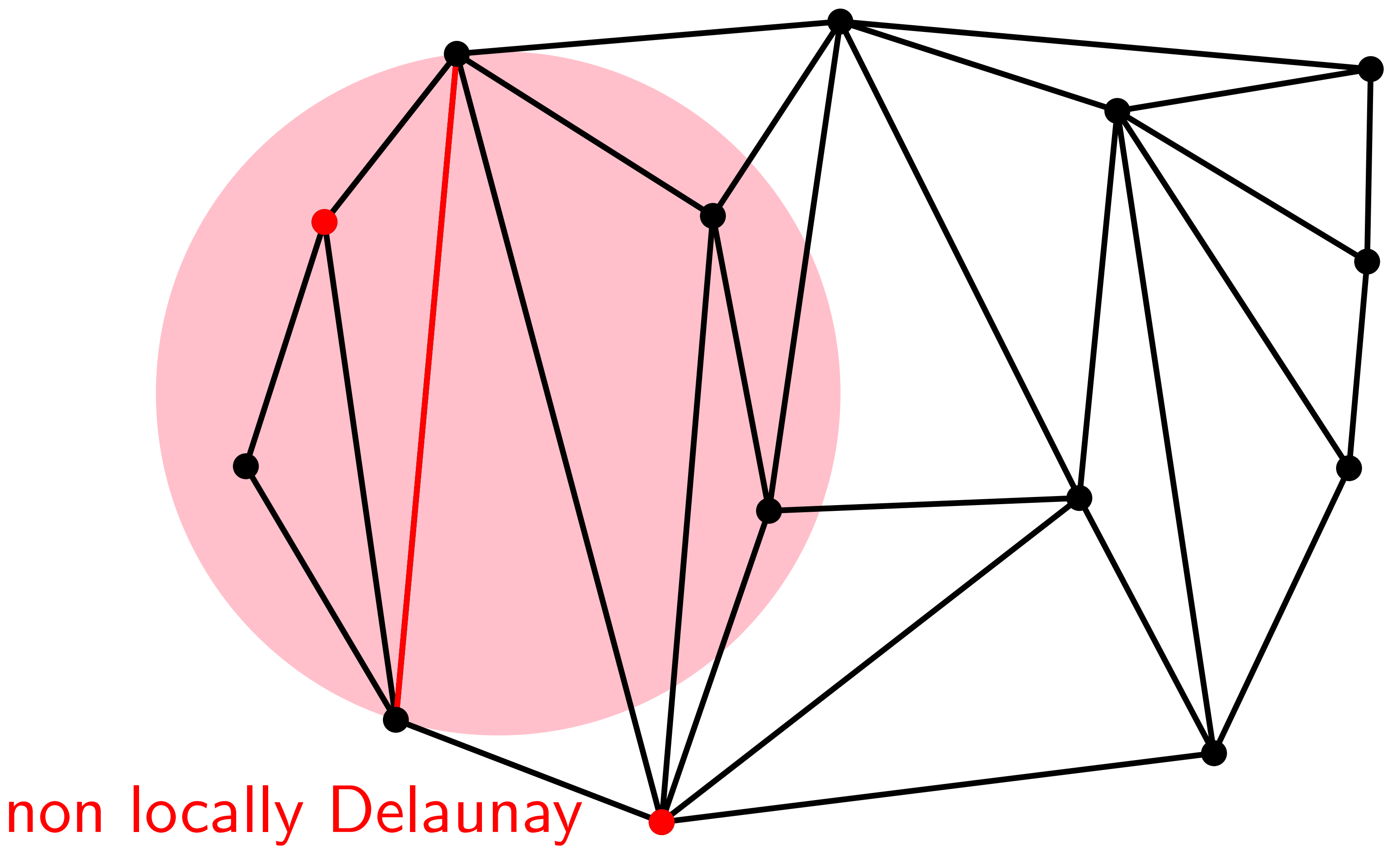
Delaunay Triangulation: Diagonal flipping



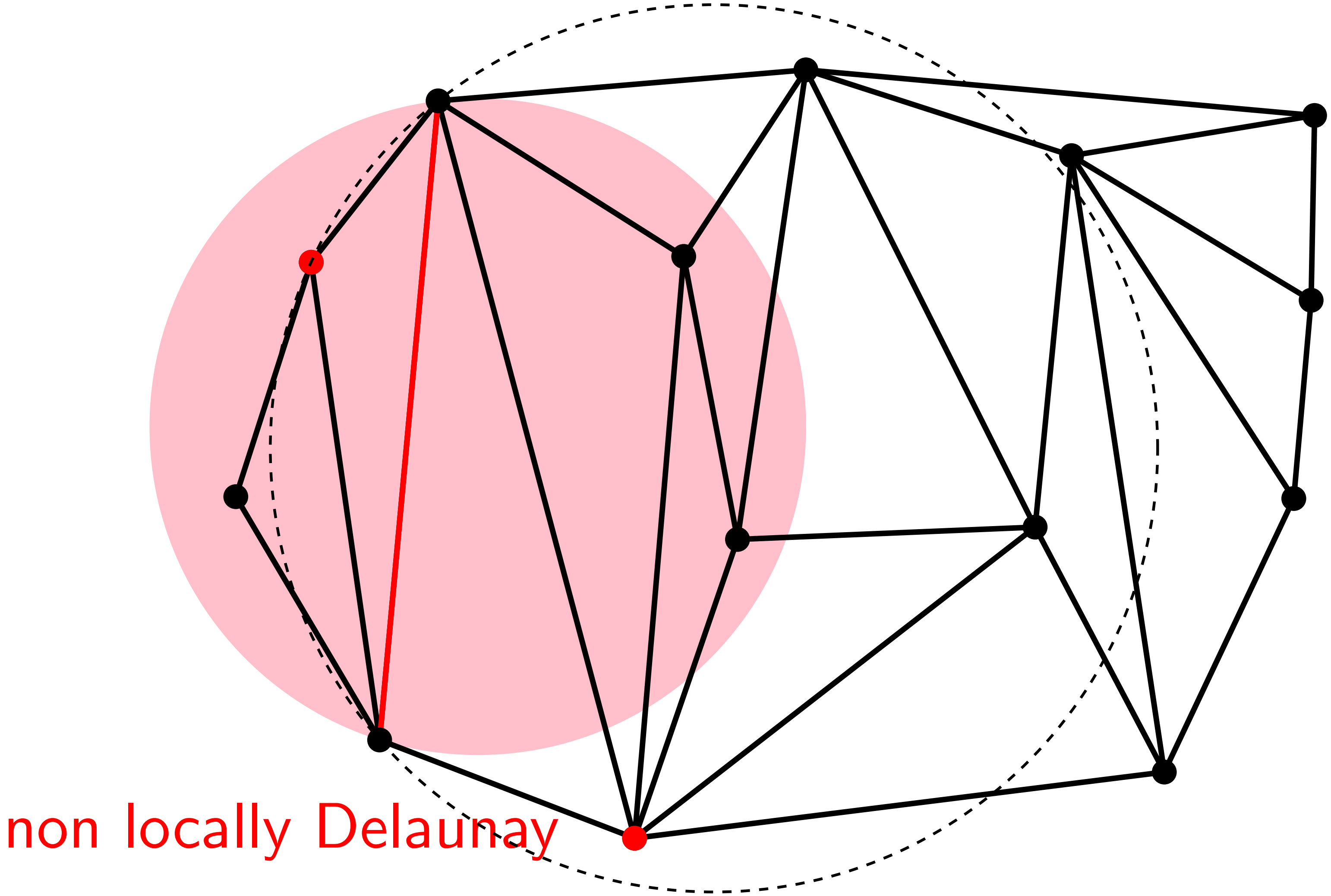
Delaunay Triangulation: Diagonal flipping



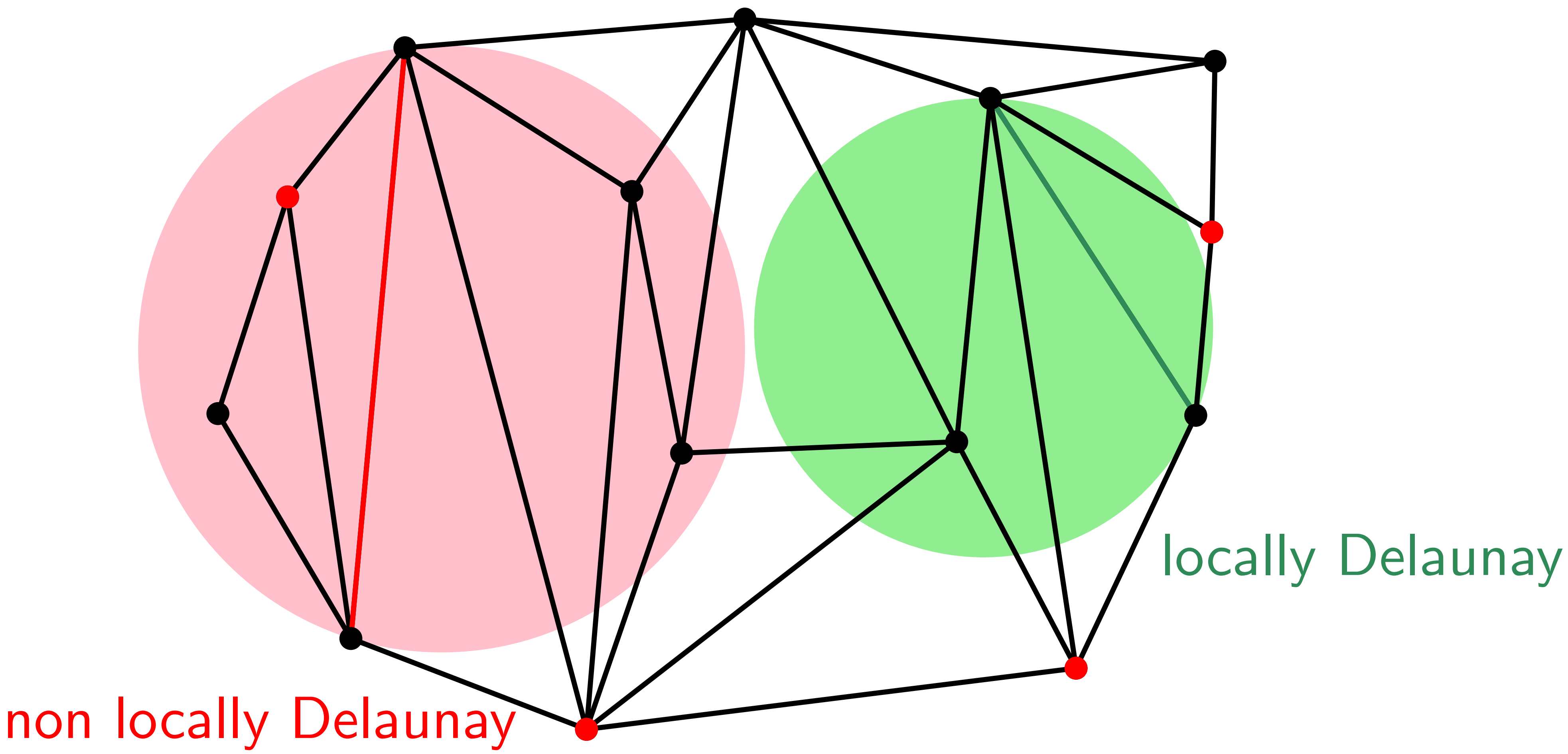
Delaunay Triangulation: Diagonal flipping



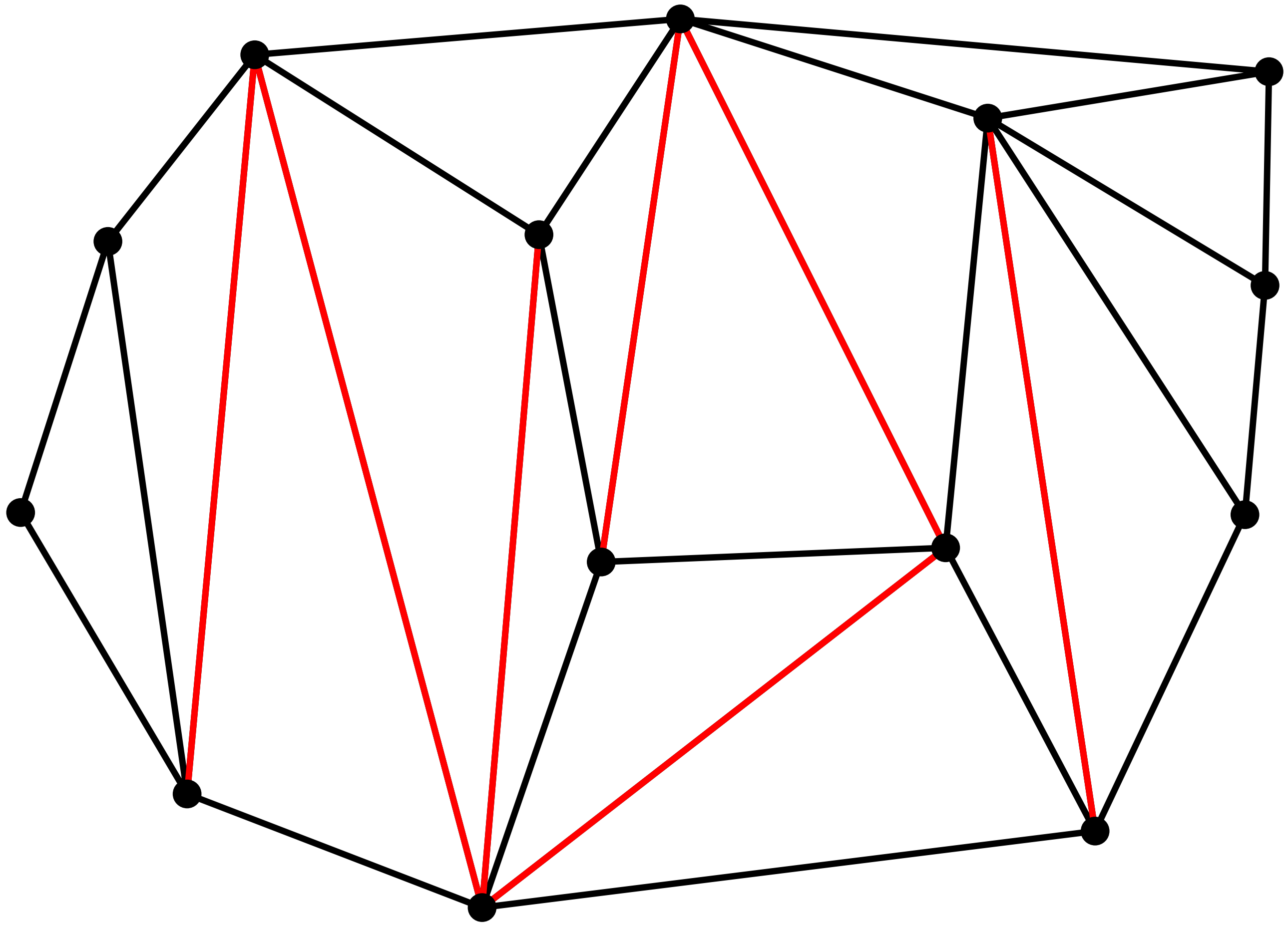
Delaunay Triangulation: Diagonal flipping



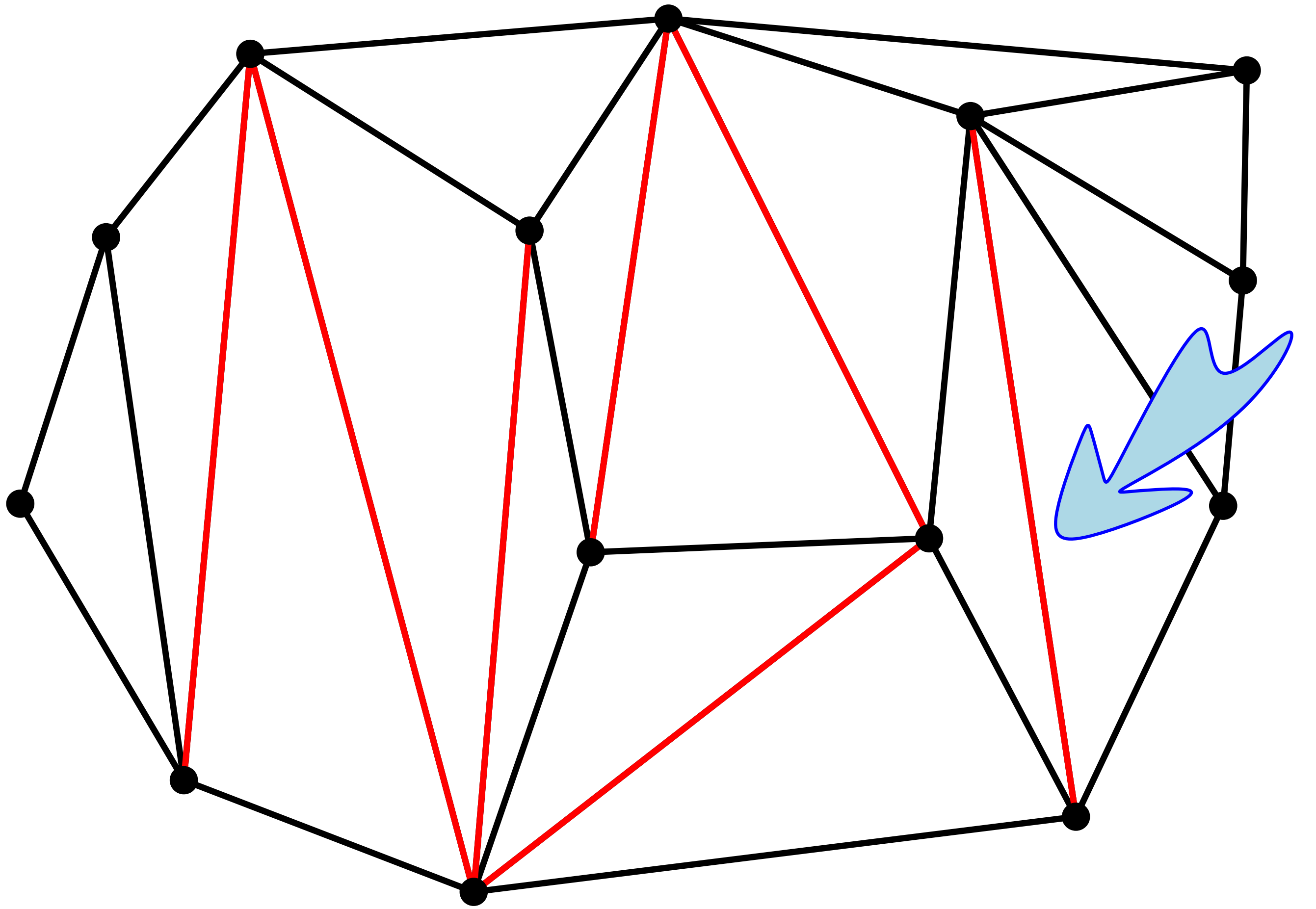
Delaunay Triangulation: Diagonal flipping



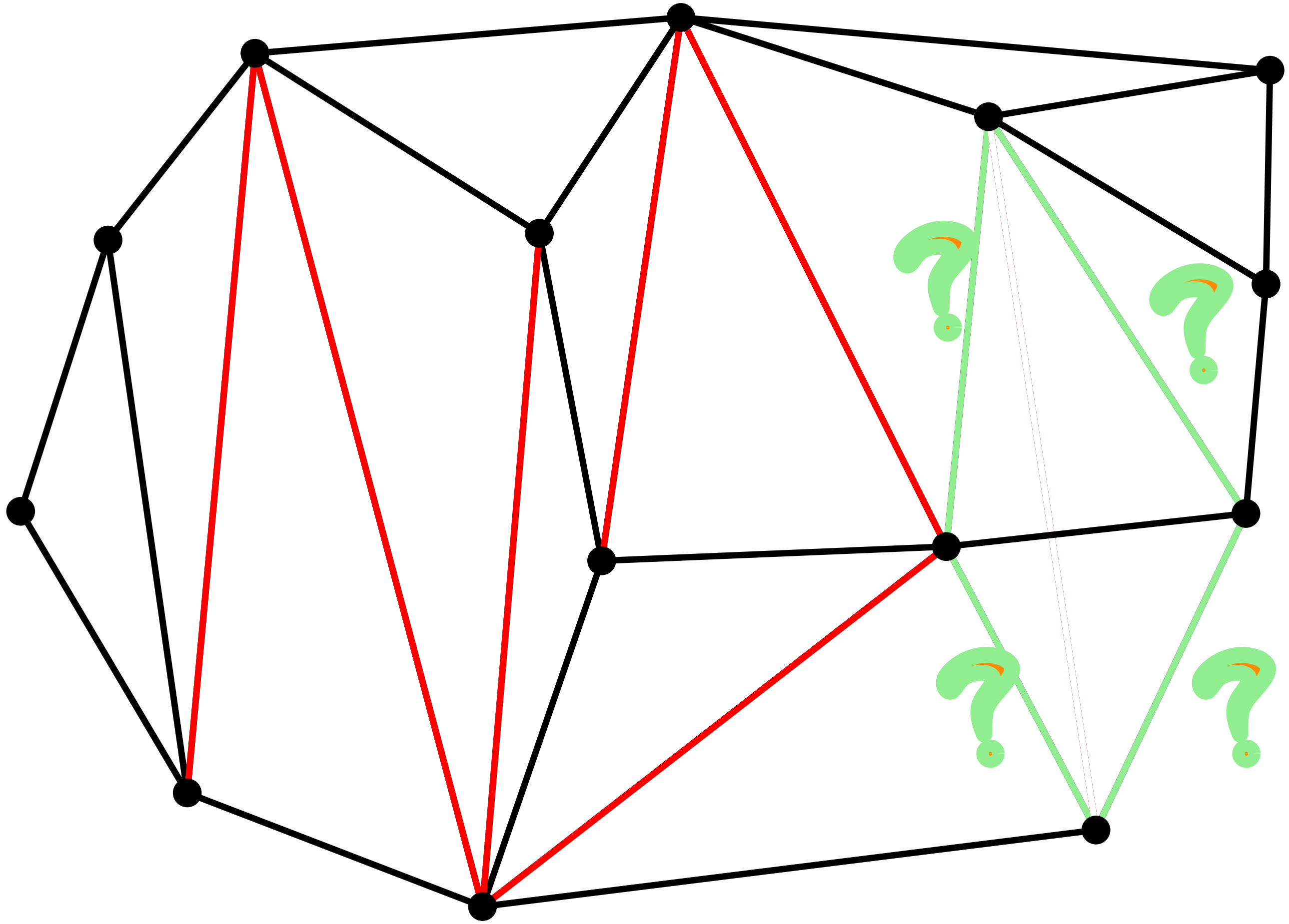
Delaunay Triangulation: Diagonal flipping



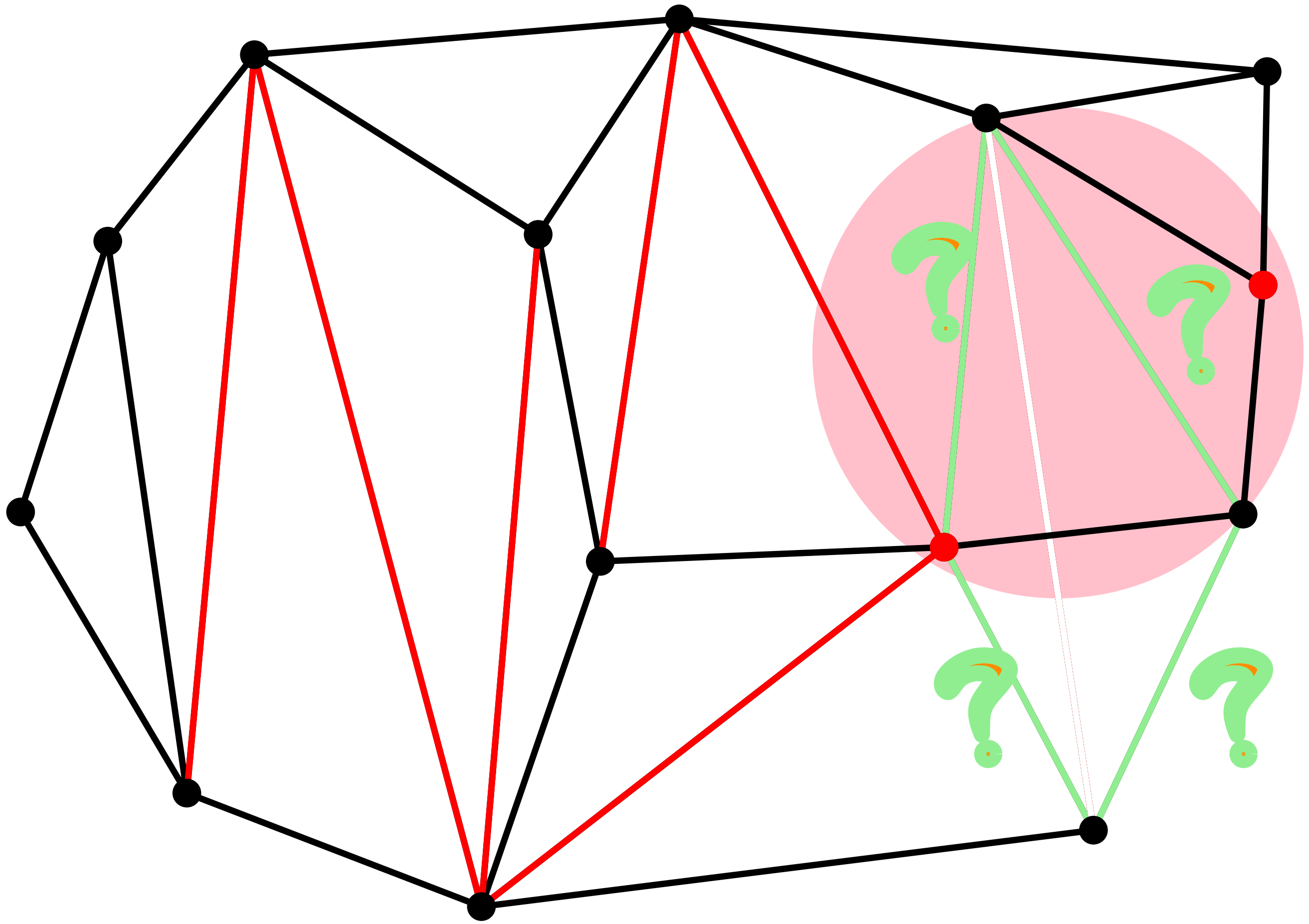
Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping

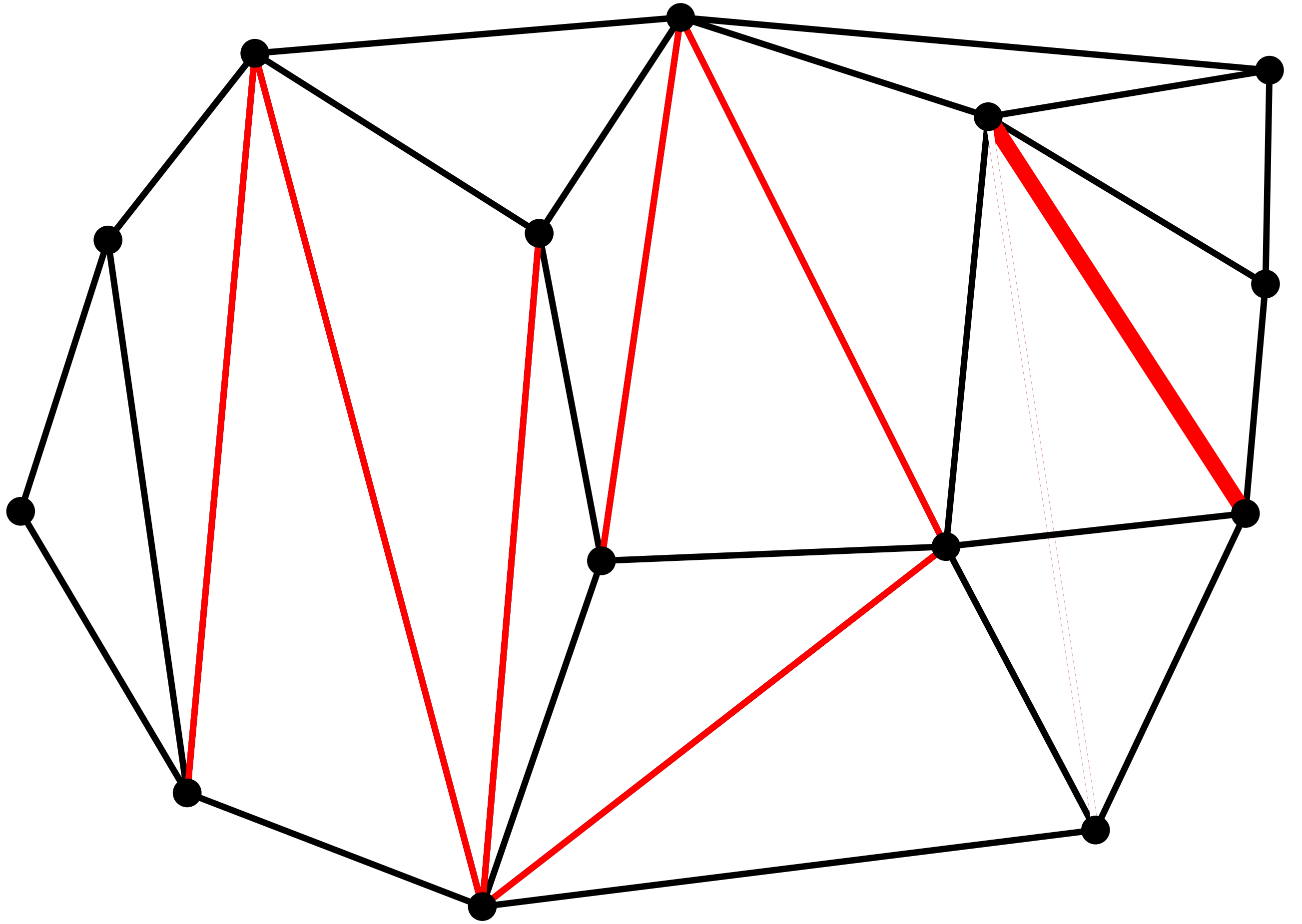


Delaunay Triangulation: Diagonal flipping

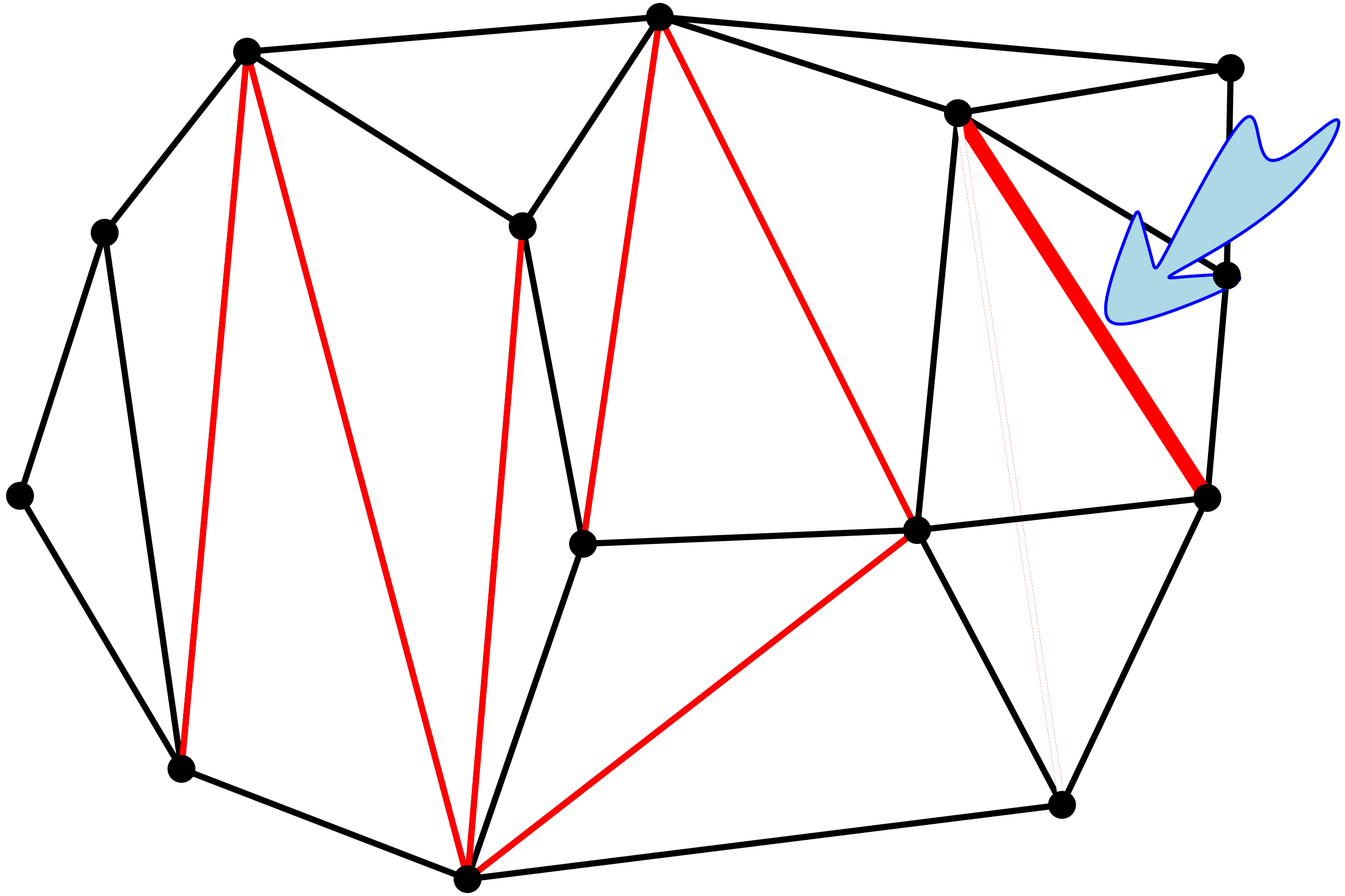


check edges of quadrilateral

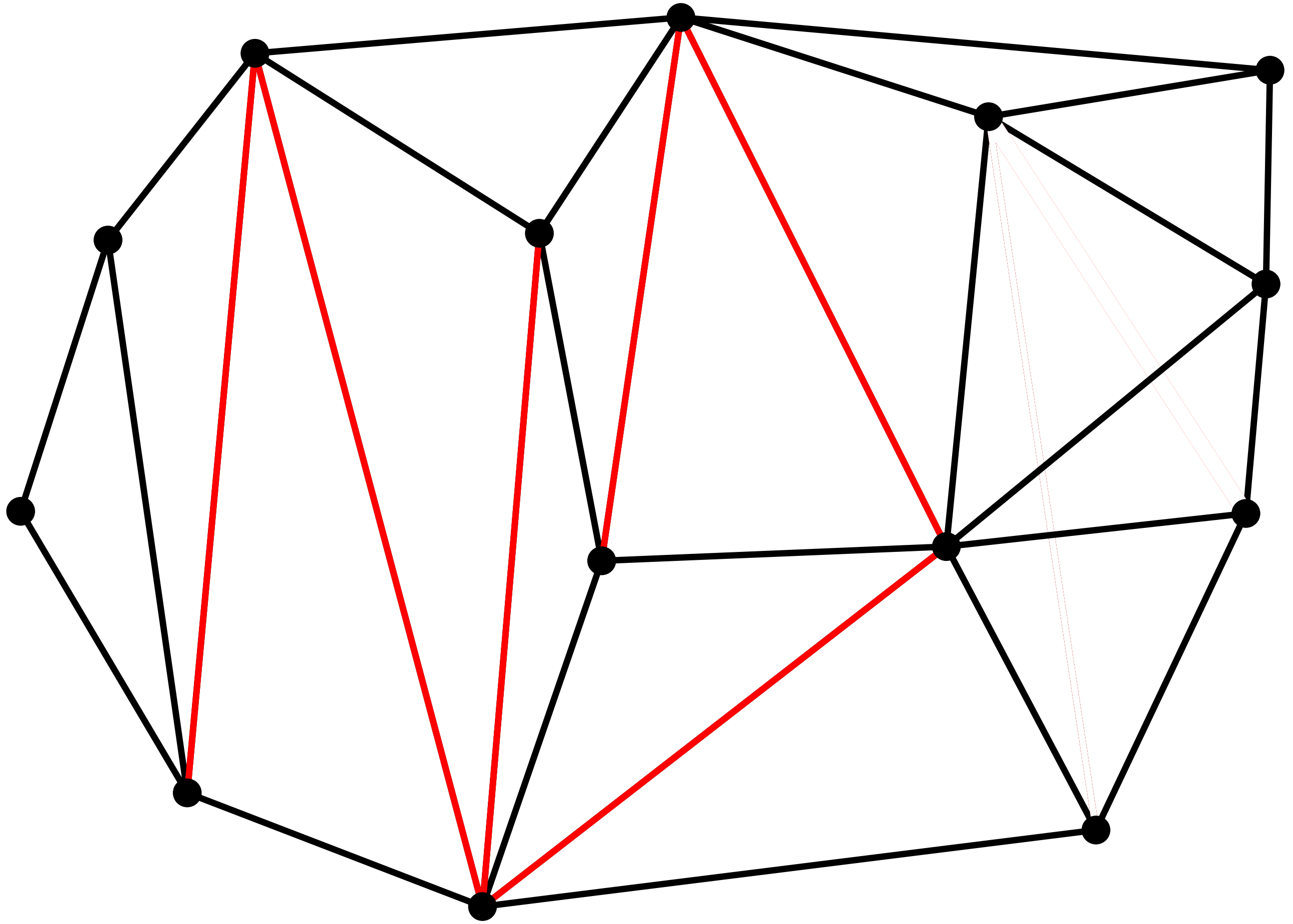
Delaunay Triangulation: Diagonal flipping



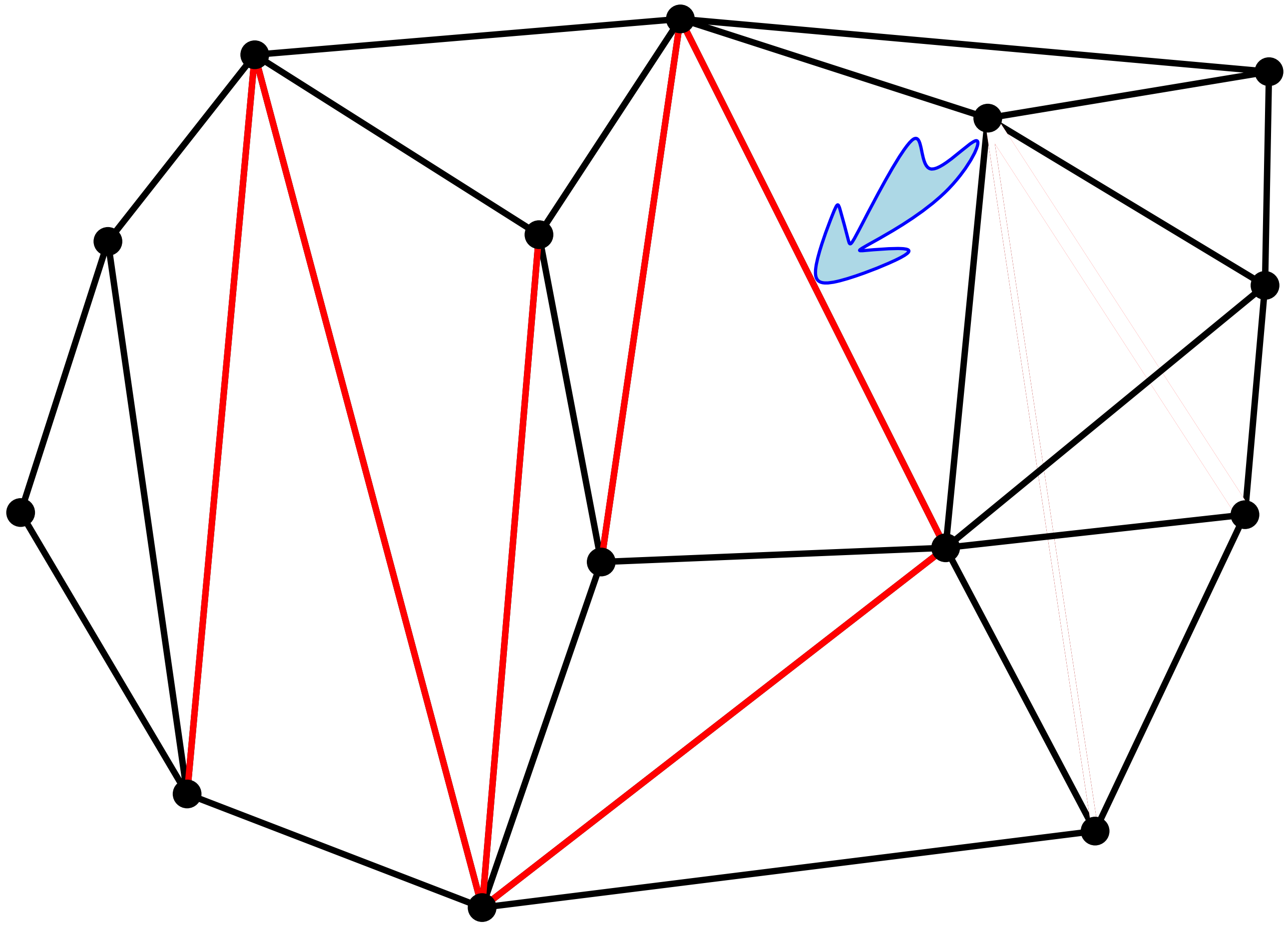
Delaunay Triangulation: Diagonal flipping



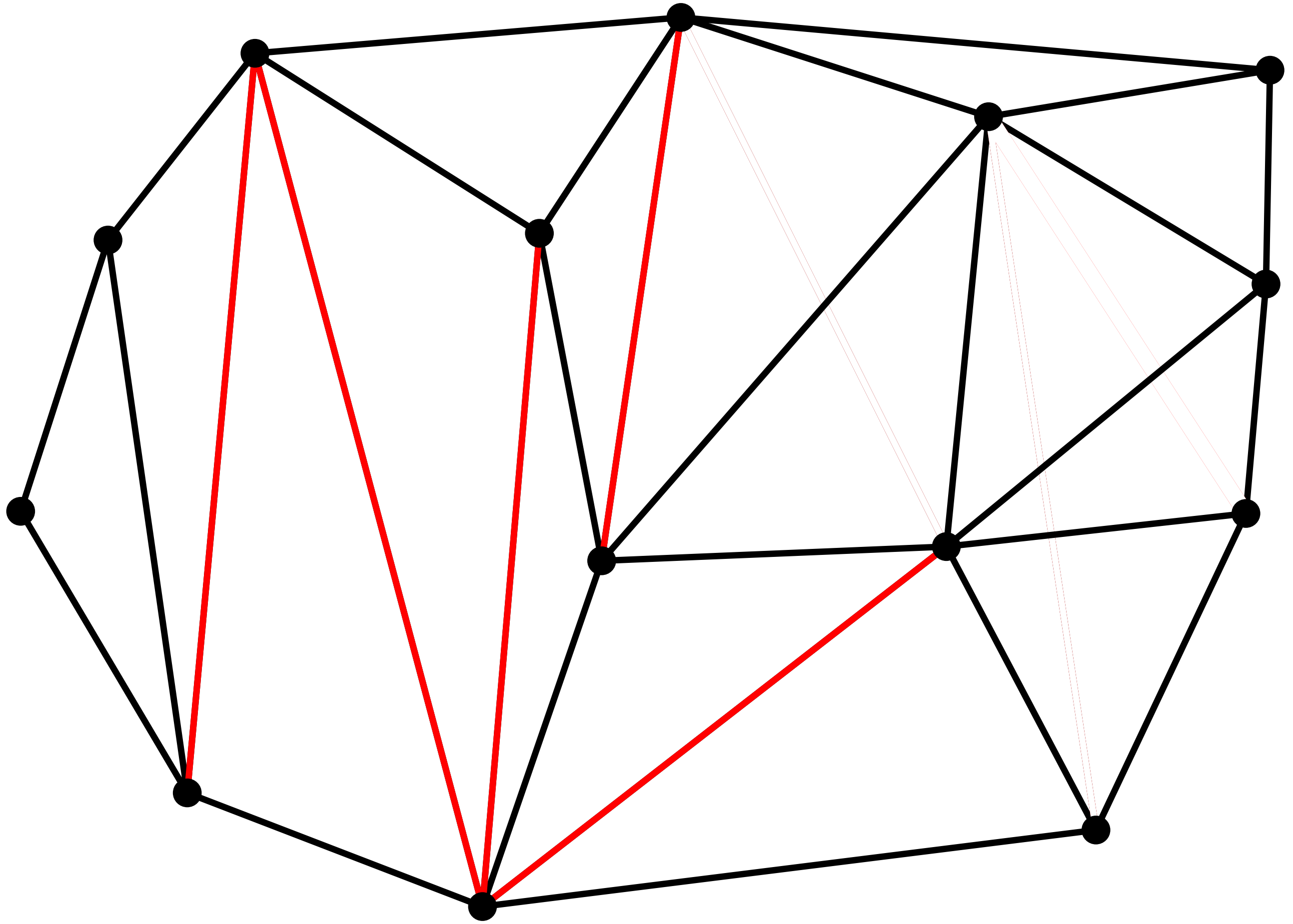
Delaunay Triangulation: Diagonal flipping



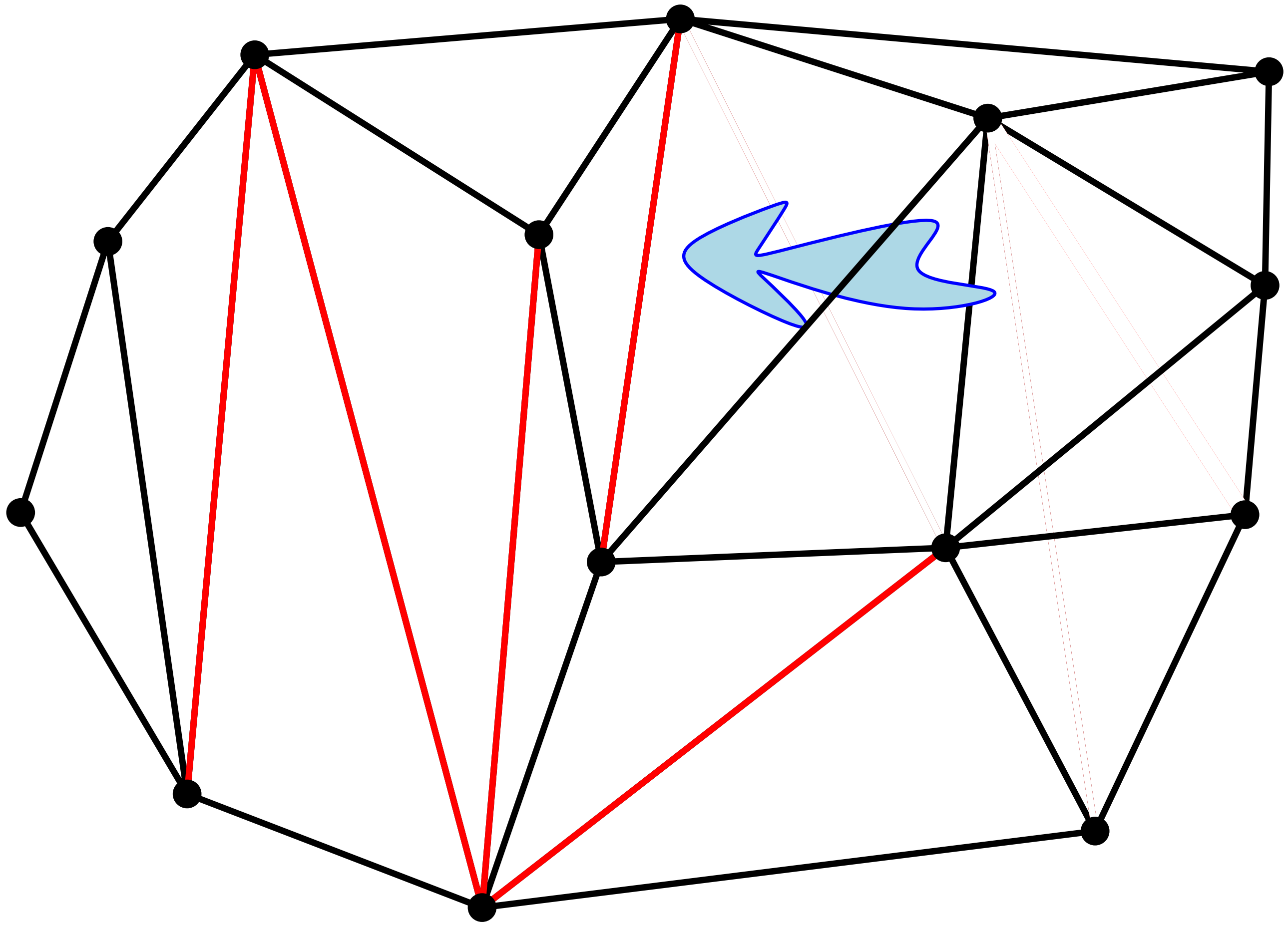
Delaunay Triangulation: Diagonal flipping



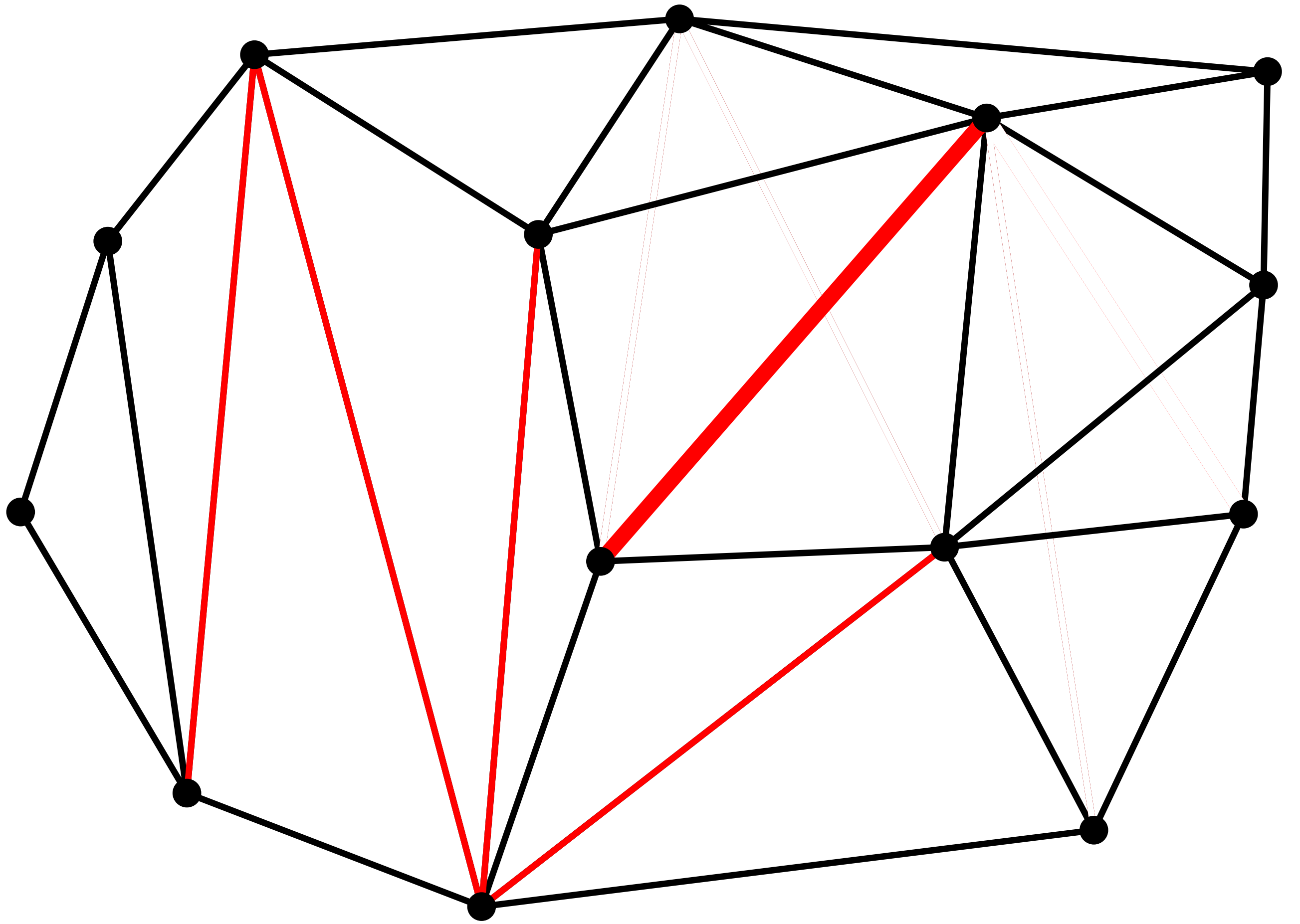
Delaunay Triangulation: Diagonal flipping



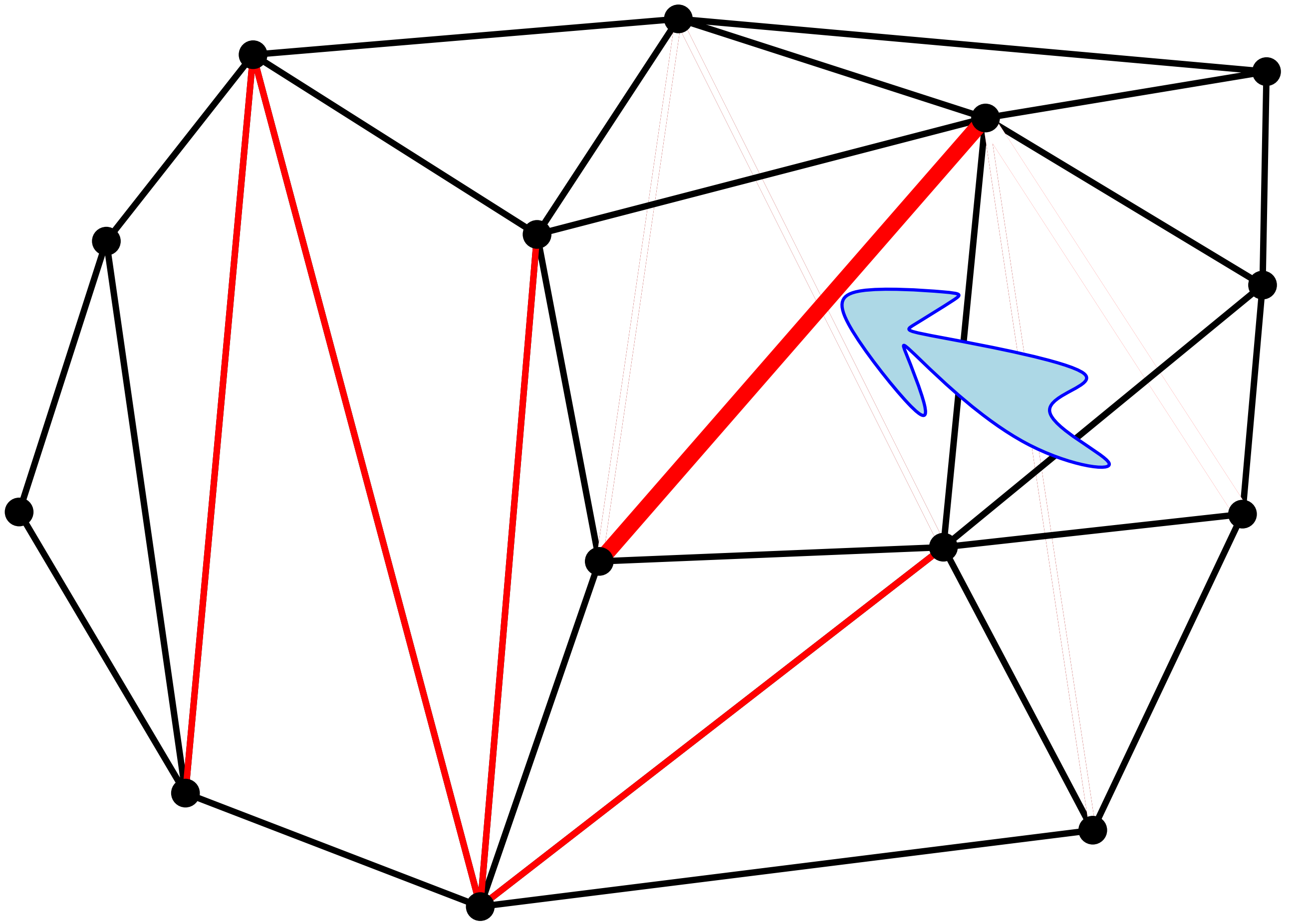
Delaunay Triangulation: Diagonal flipping



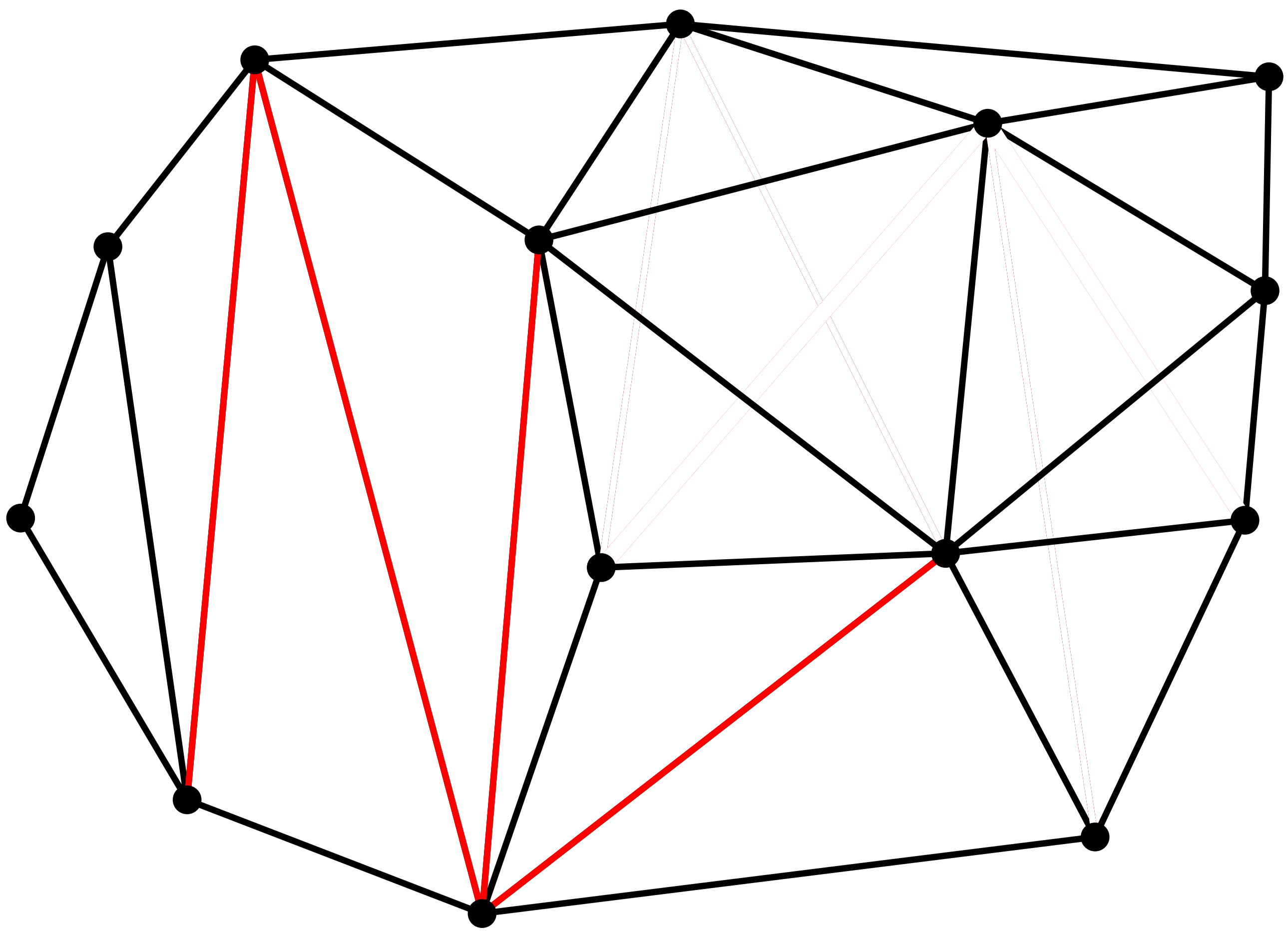
Delaunay Triangulation: Diagonal flipping



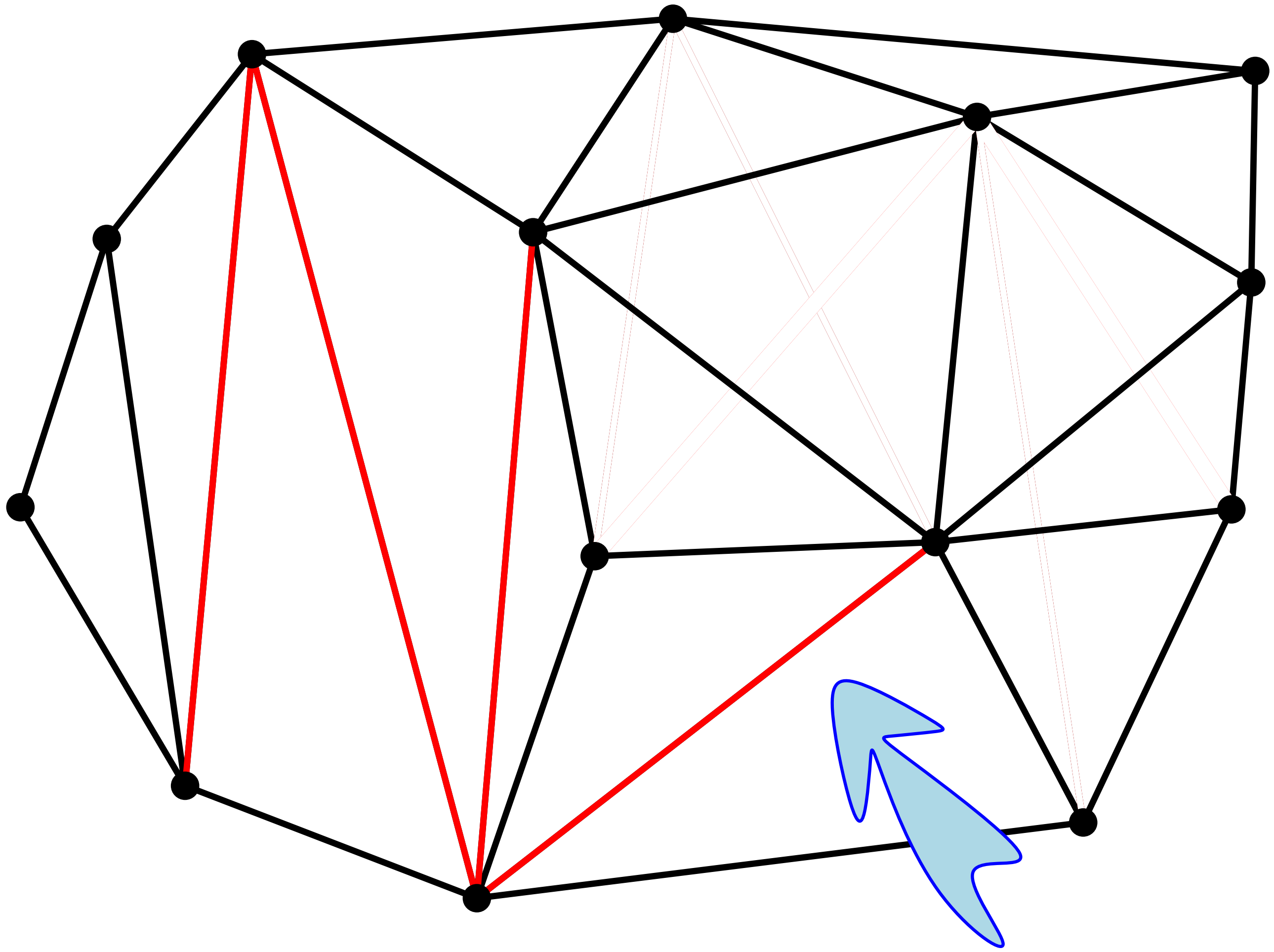
Delaunay Triangulation: Diagonal flipping



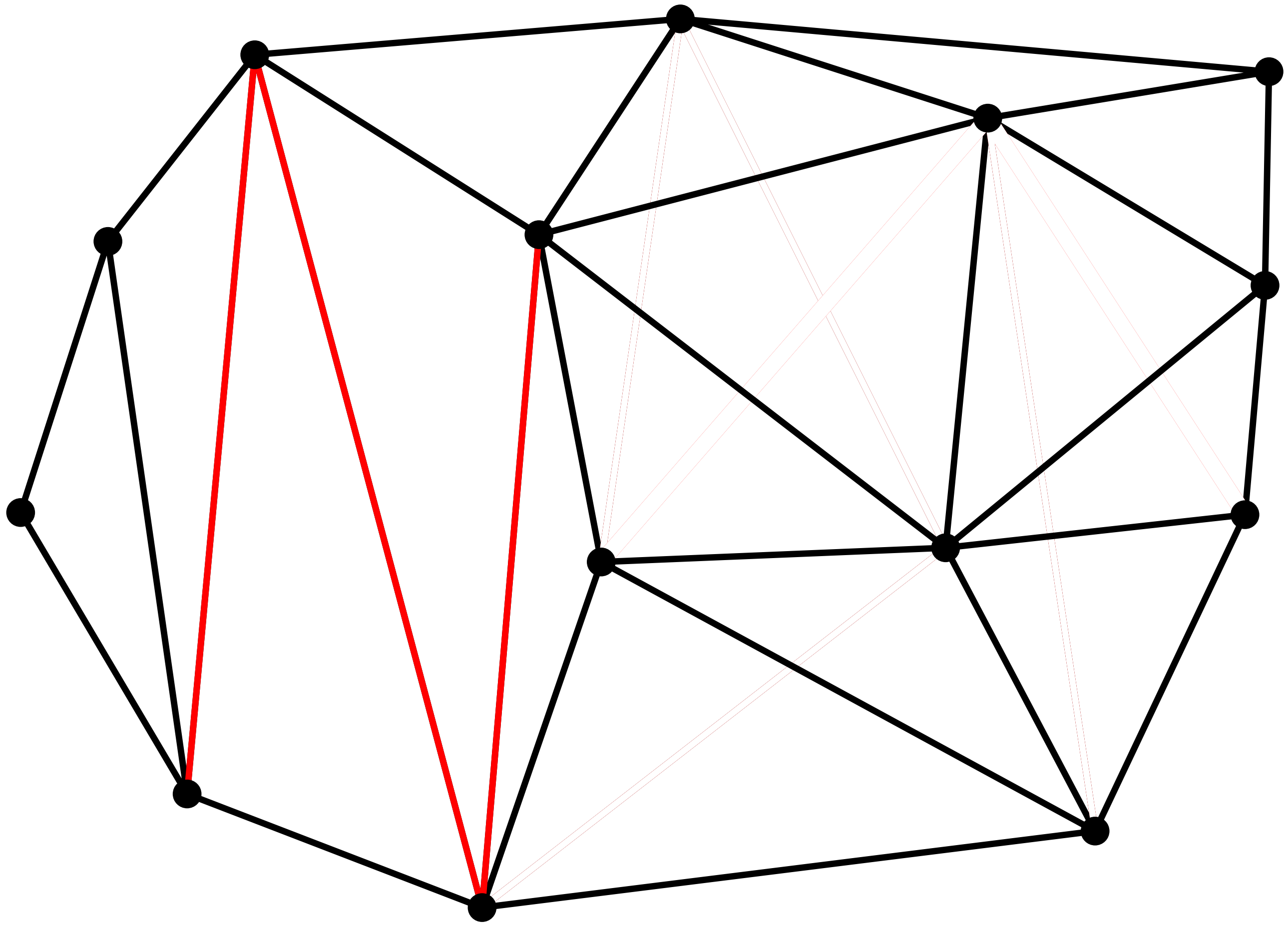
Delaunay Triangulation: Diagonal flipping



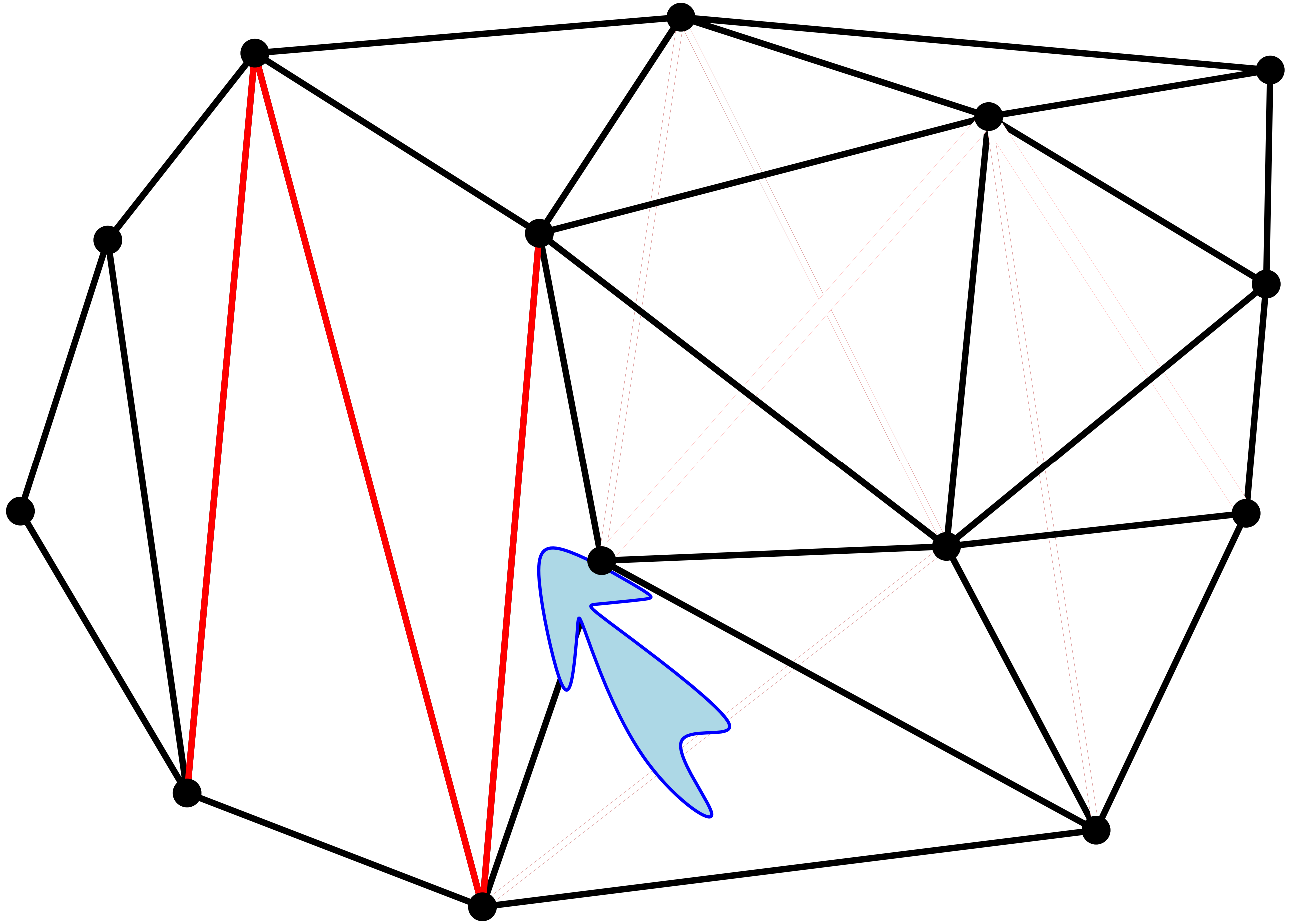
Delaunay Triangulation: Diagonal flipping



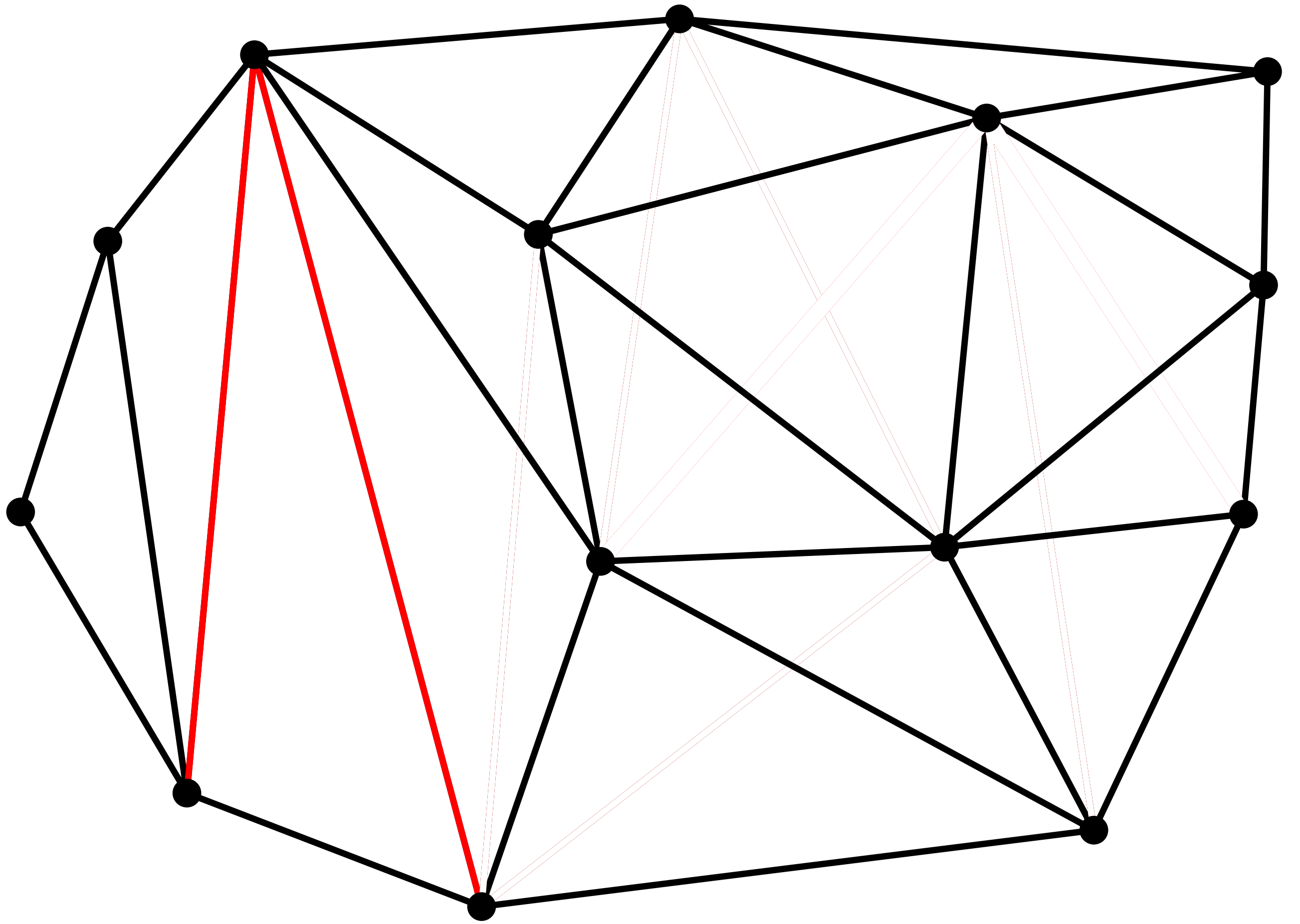
Delaunay Triangulation: Diagonal flipping



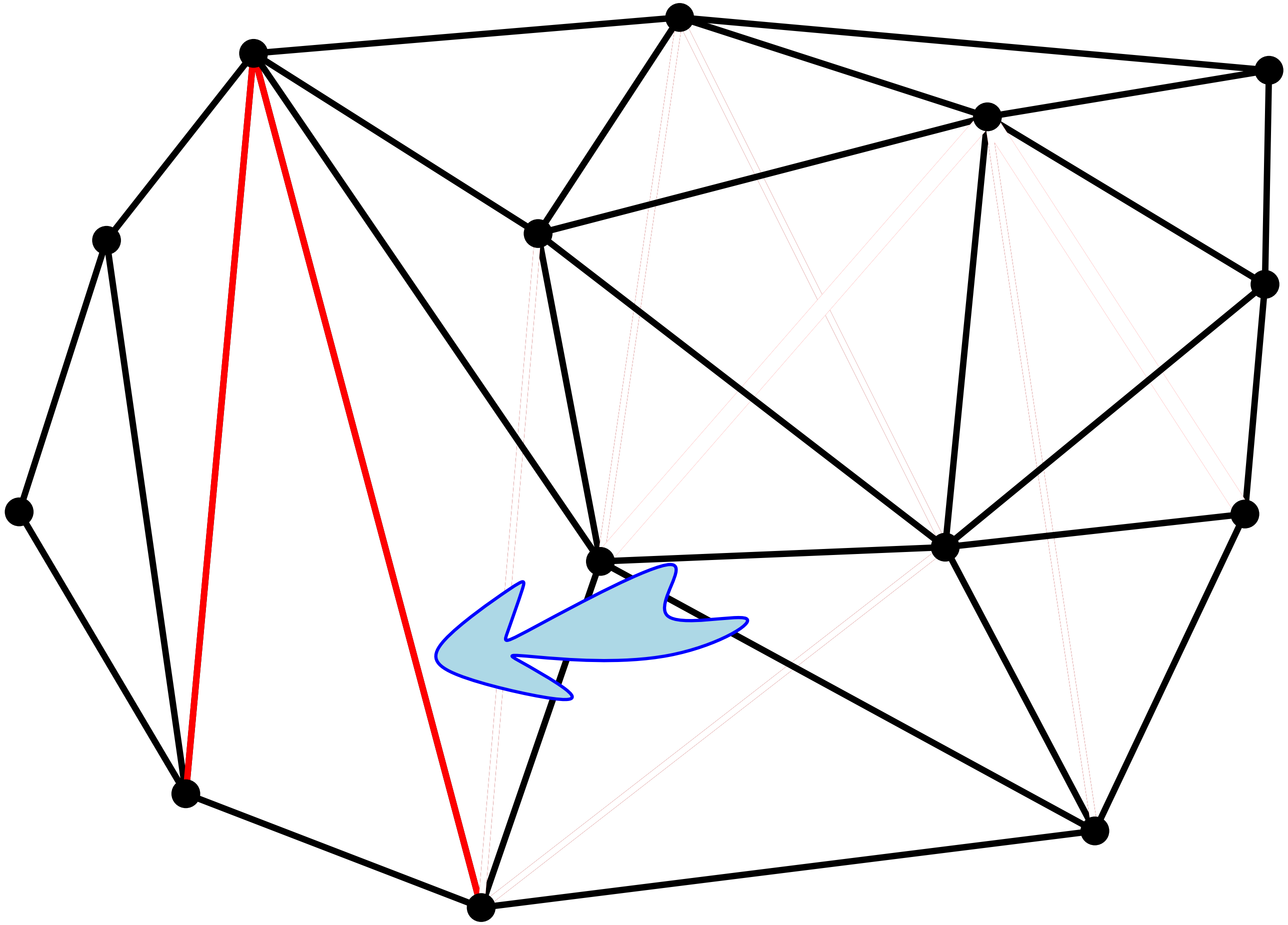
Delaunay Triangulation: Diagonal flipping



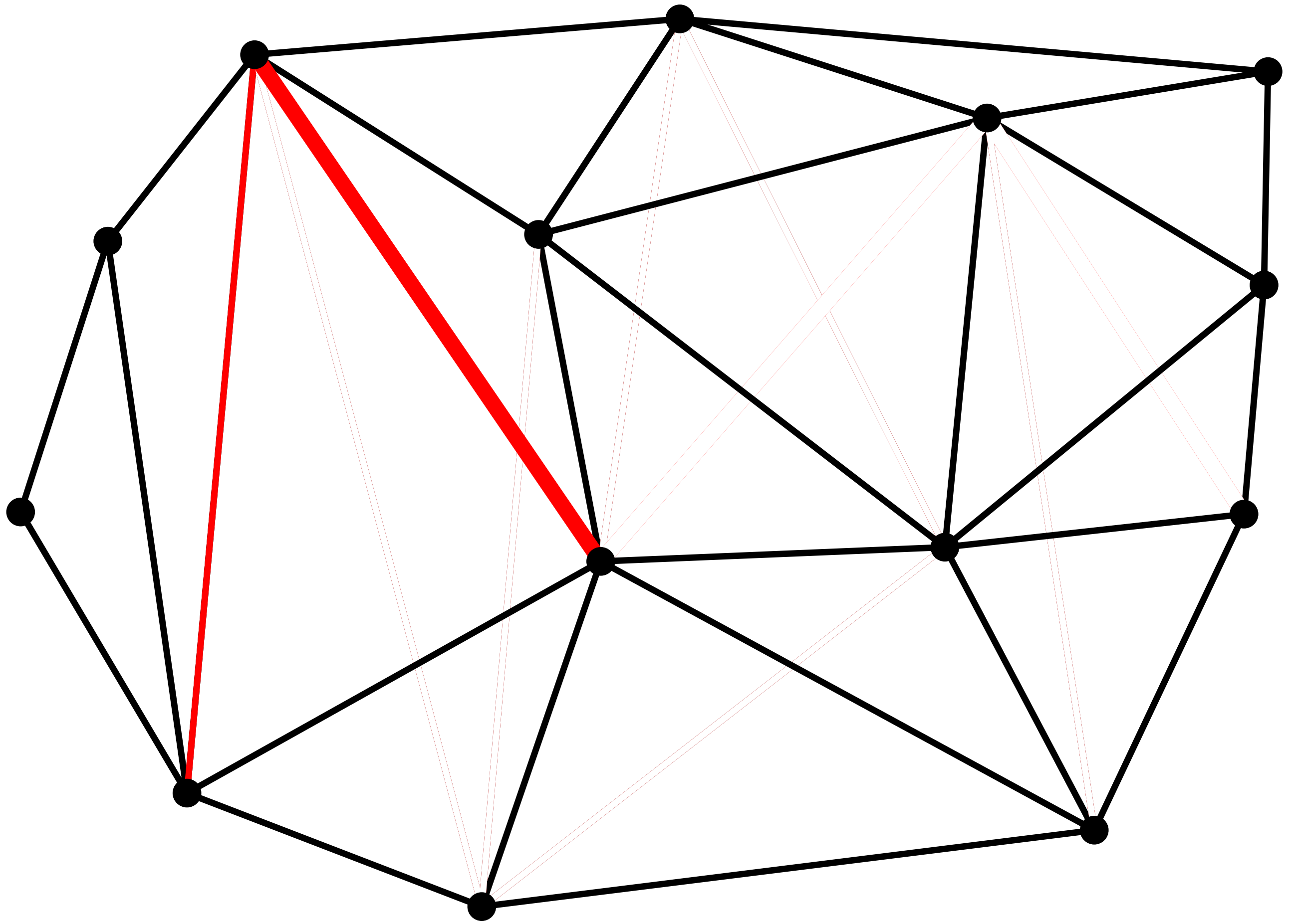
Delaunay Triangulation: Diagonal flipping



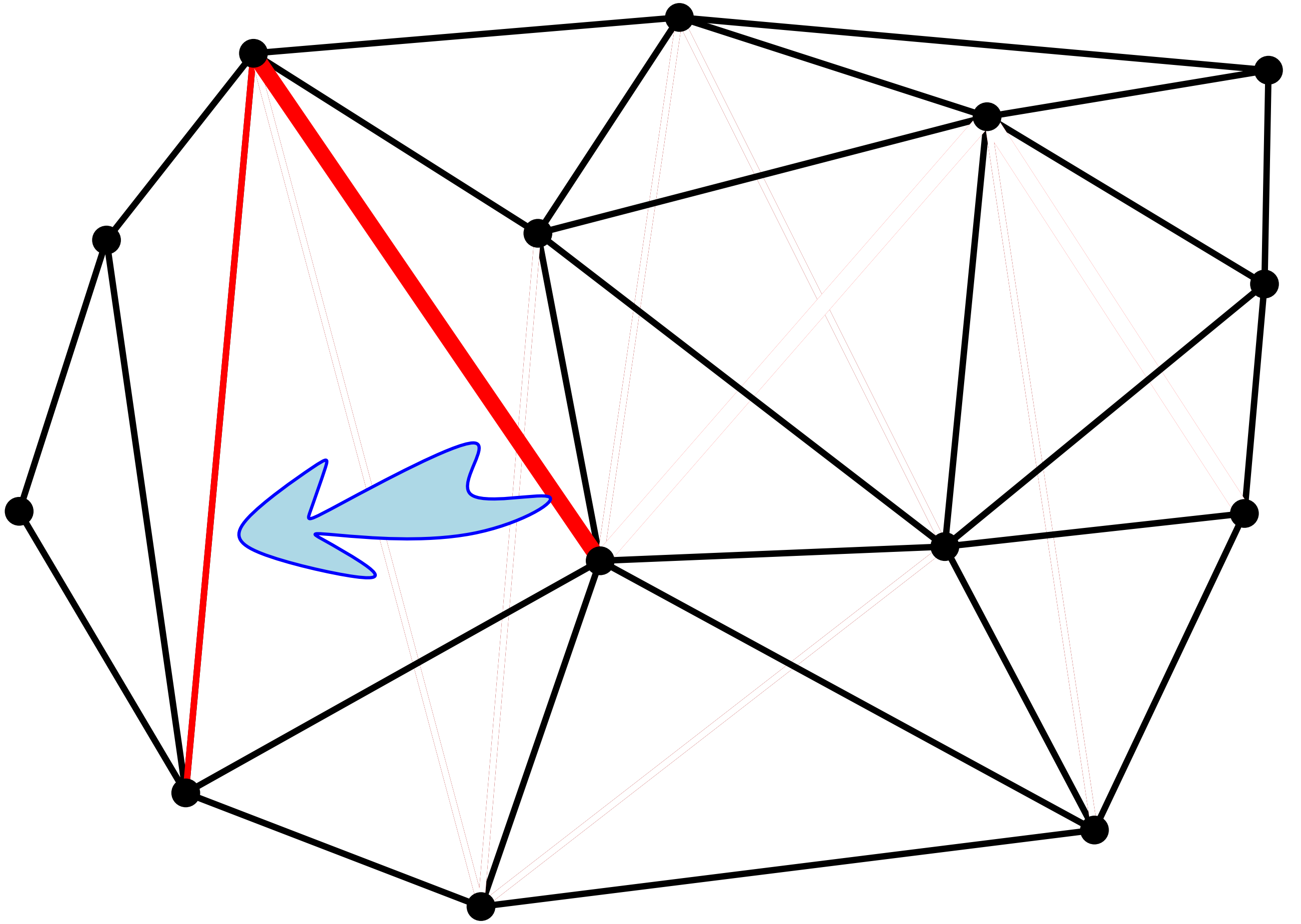
Delaunay Triangulation: Diagonal flipping



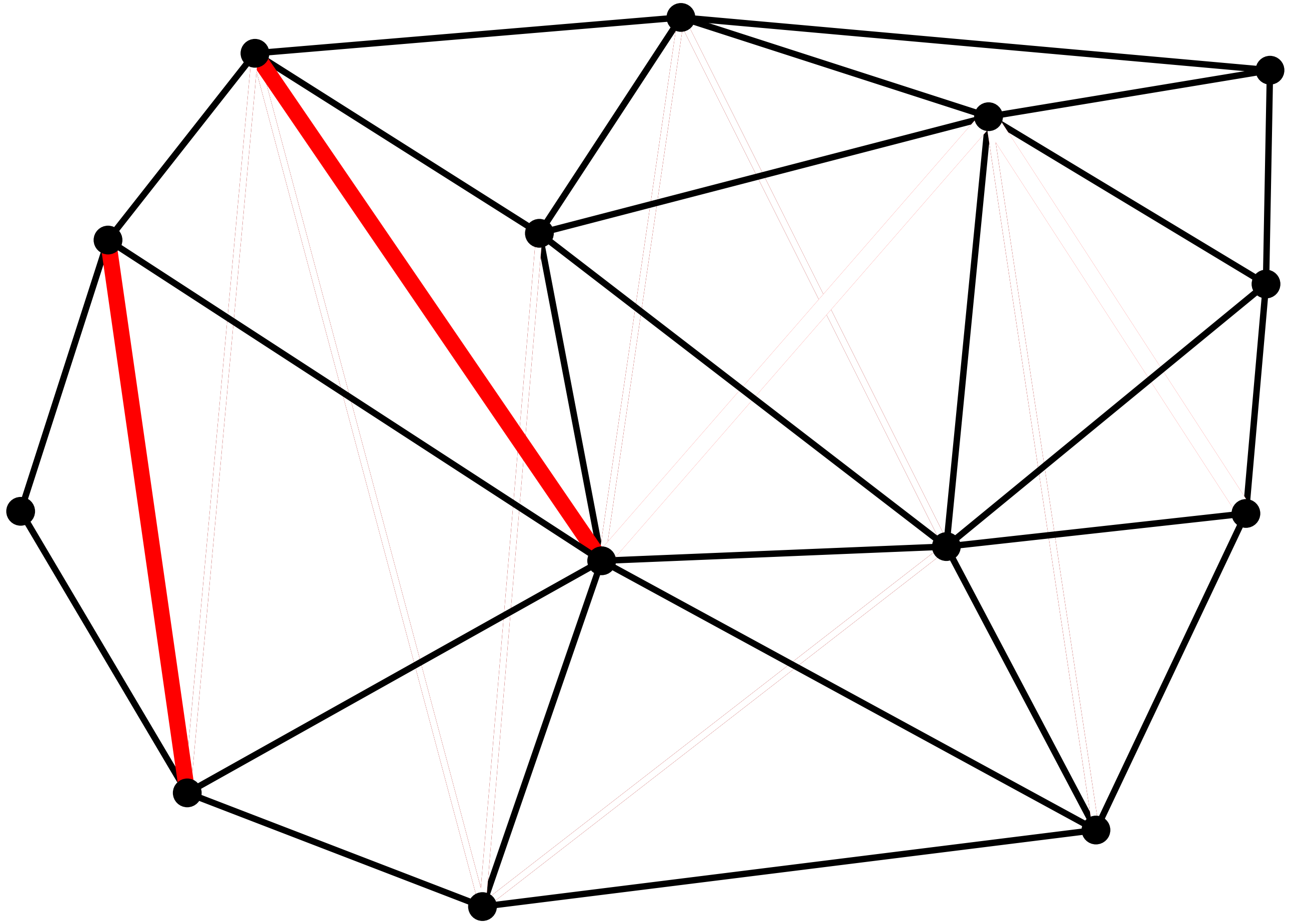
Delaunay Triangulation: Diagonal flipping



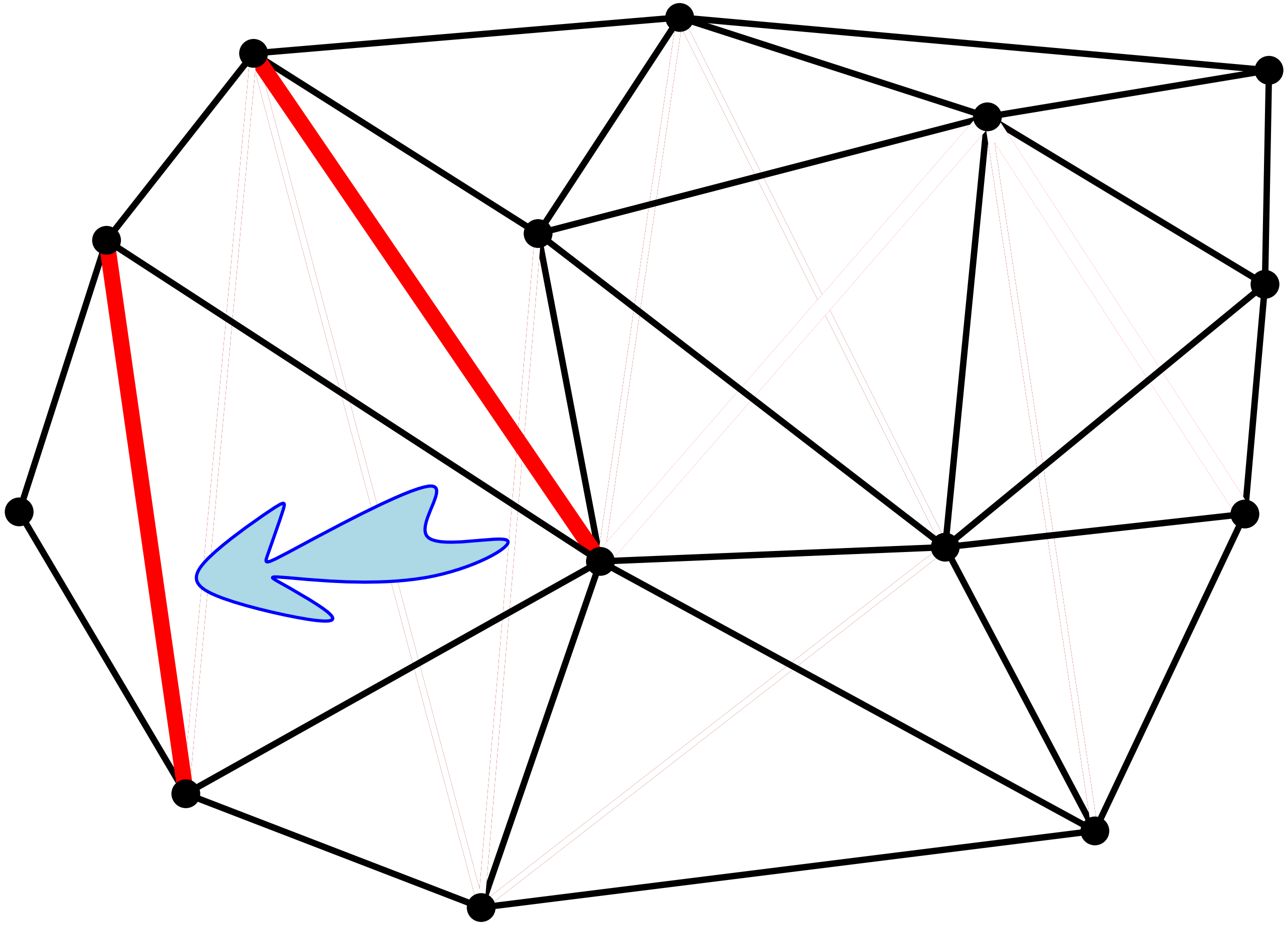
Delaunay Triangulation: Diagonal flipping



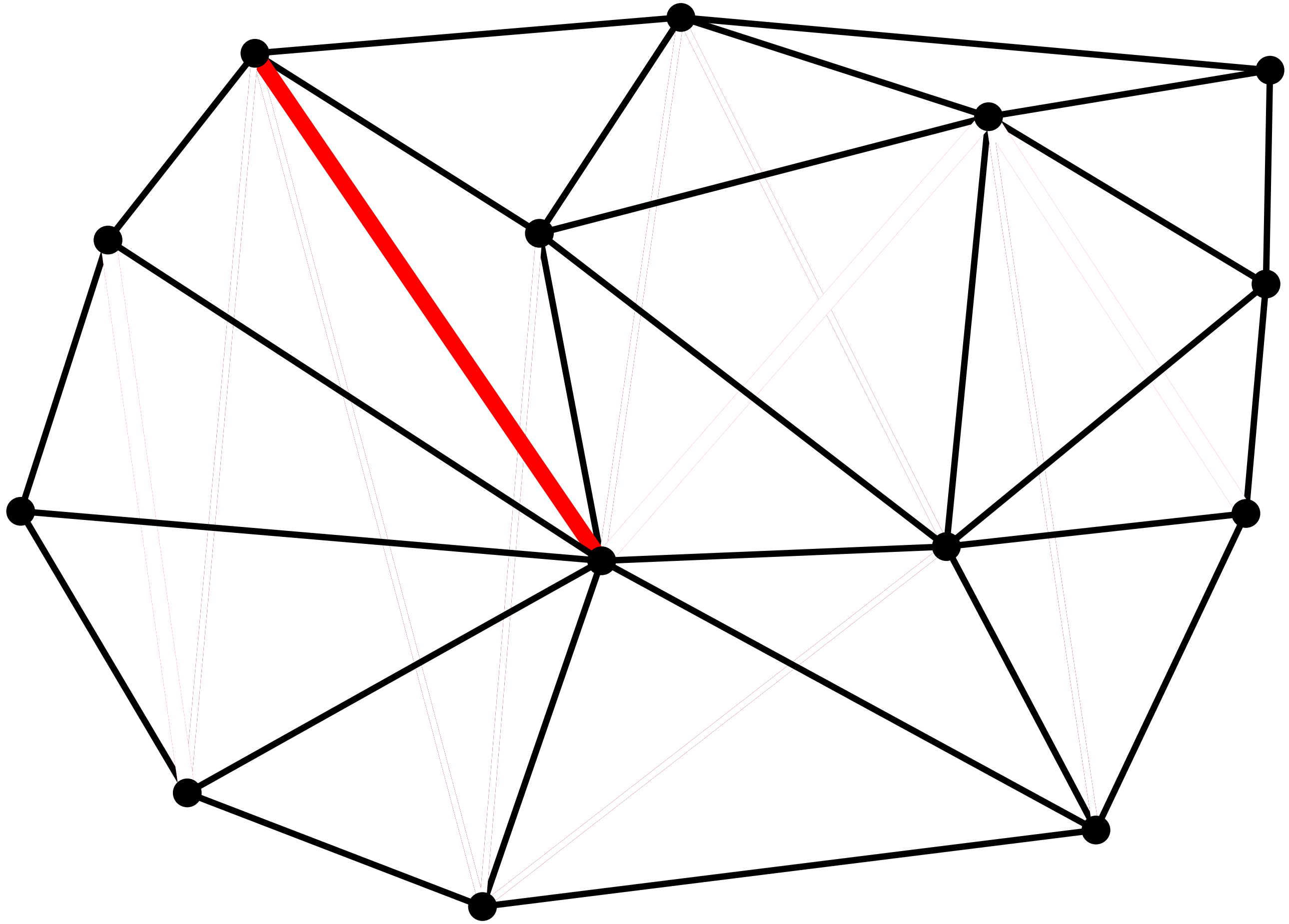
Delaunay Triangulation: Diagonal flipping



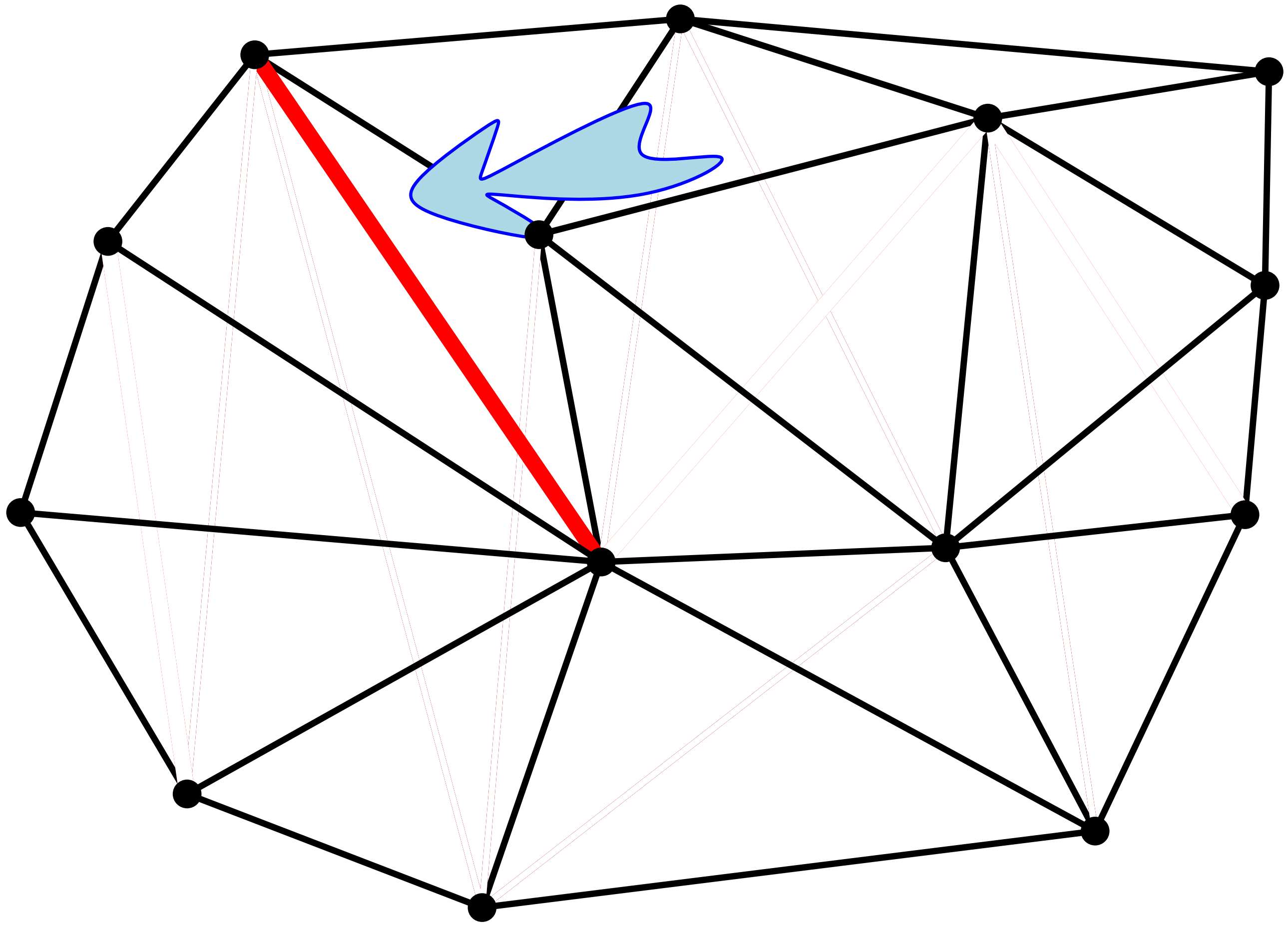
Delaunay Triangulation: Diagonal flipping



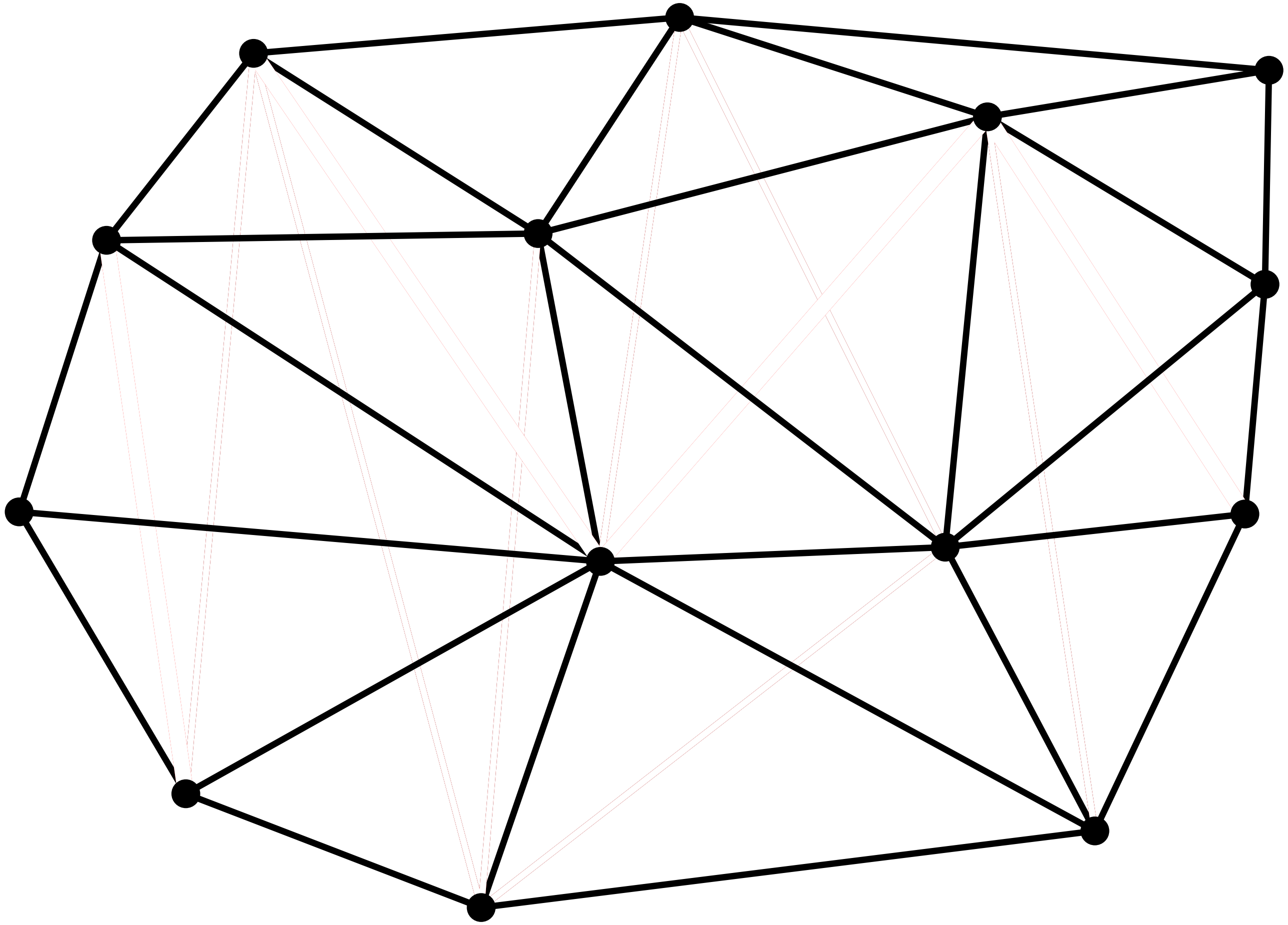
Delaunay Triangulation: Diagonal flipping



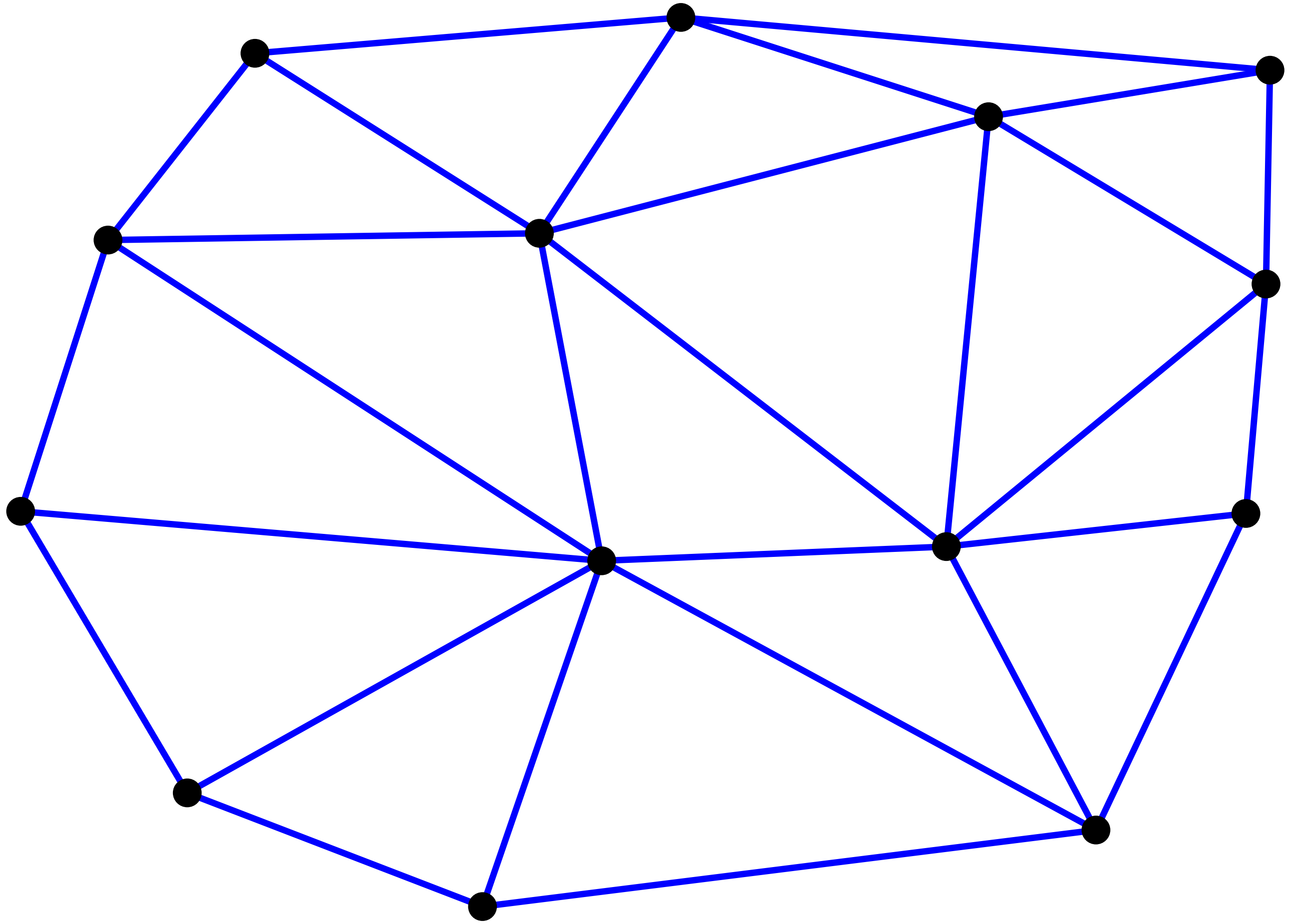
Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



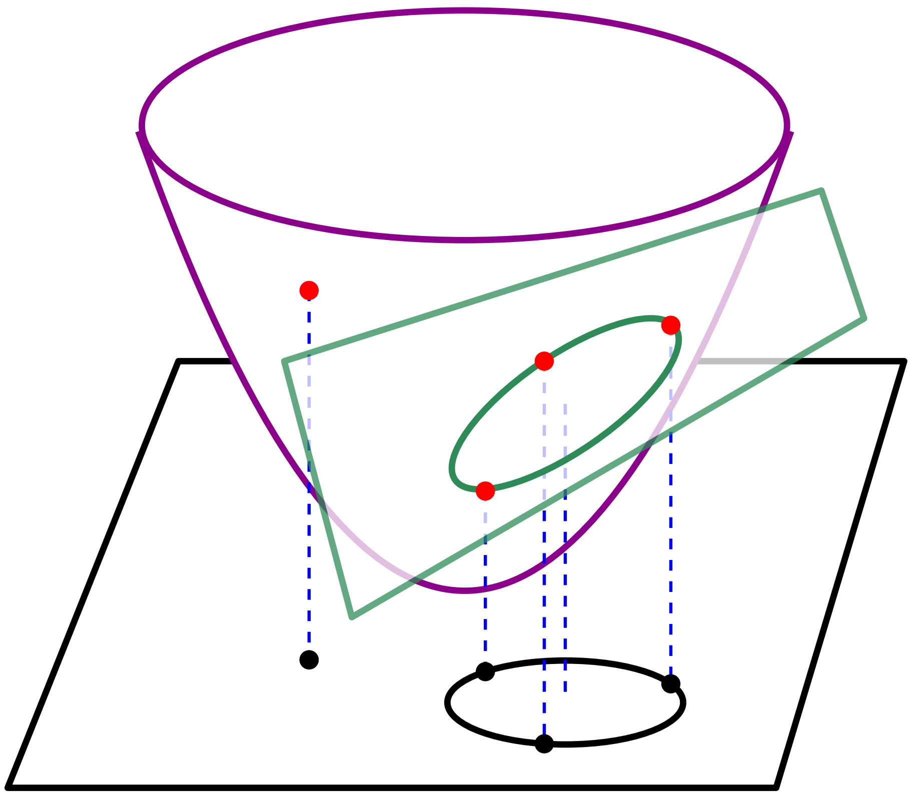
Delaunay is obtained

Delaunay Triangulation: Diagonal flipping

Complexity ?

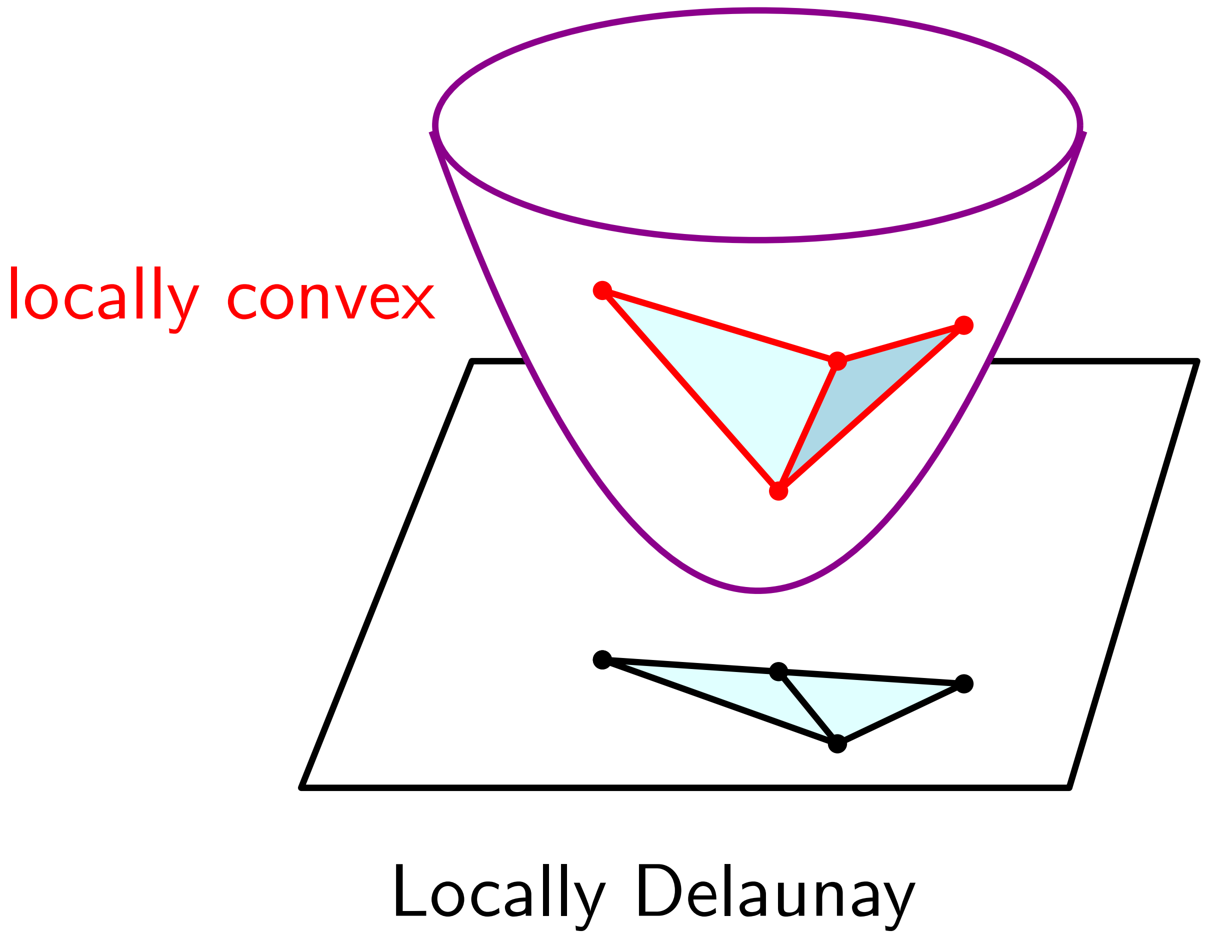
Delaunay Triangulation: Diagonal flipping

Complexity ?



Delaunay Triangulation: Diagonal flipping

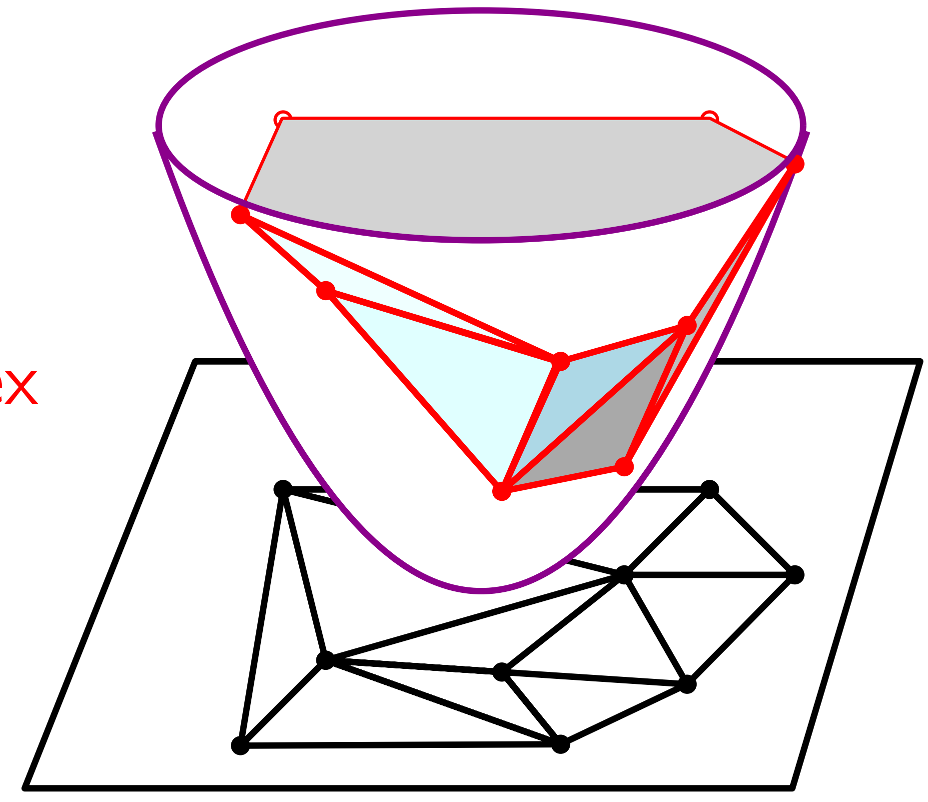
Complexity ?



Delaunay Triangulation: Diagonal flipping

Complexity ?

Convex

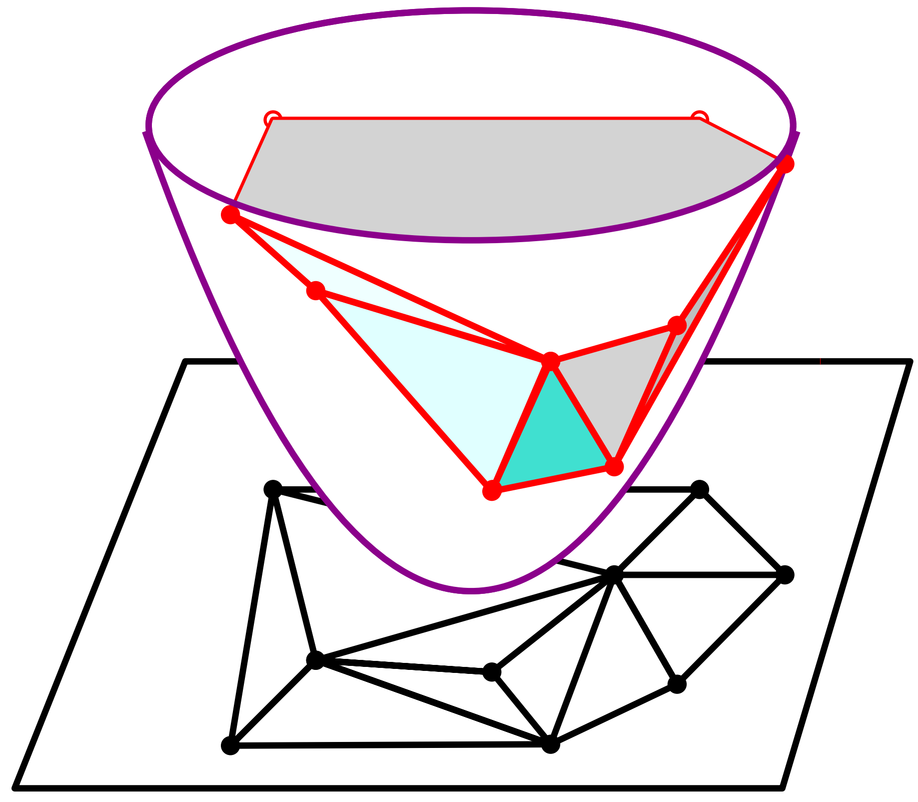


Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

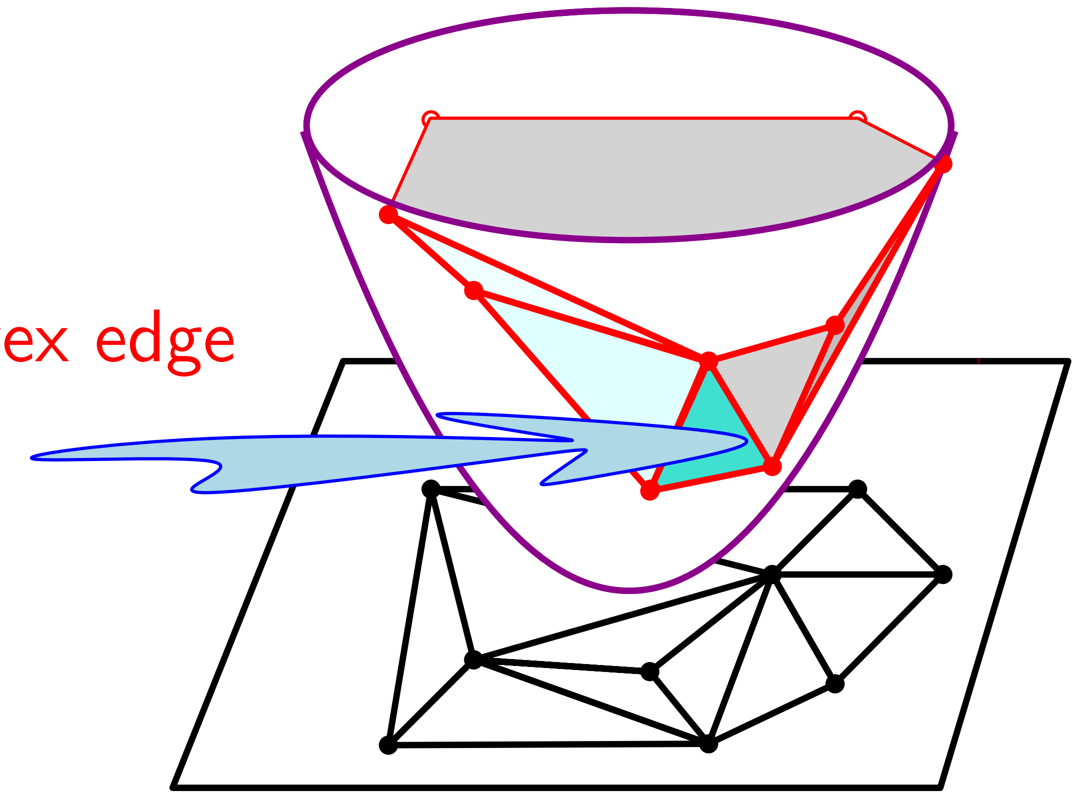


Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex edge



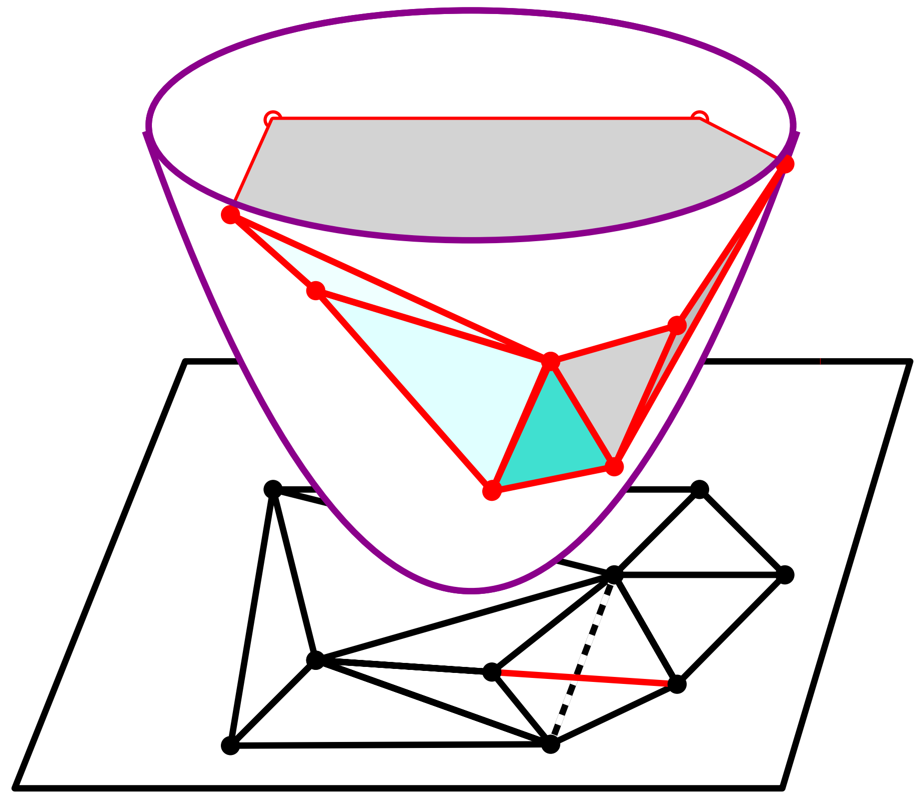
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip

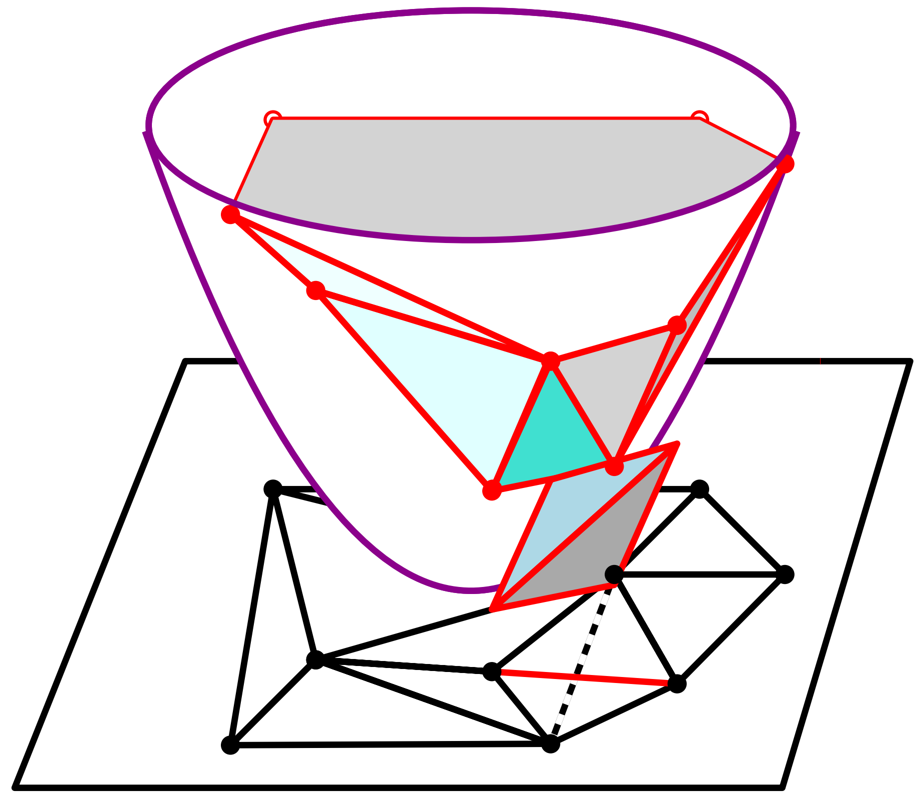


Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip

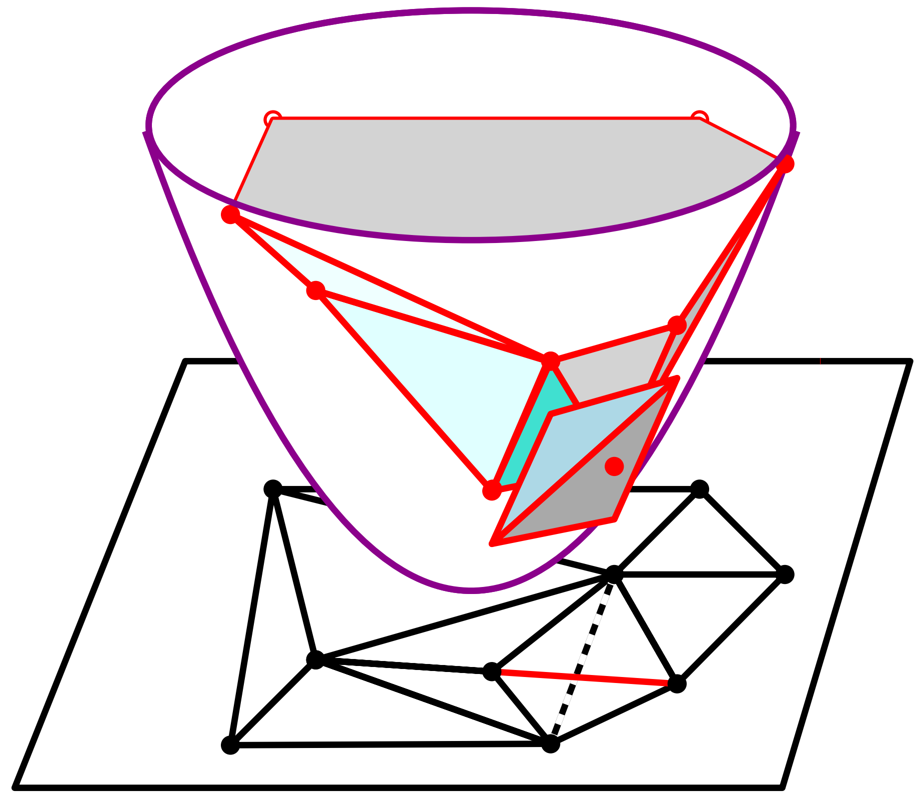


Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip

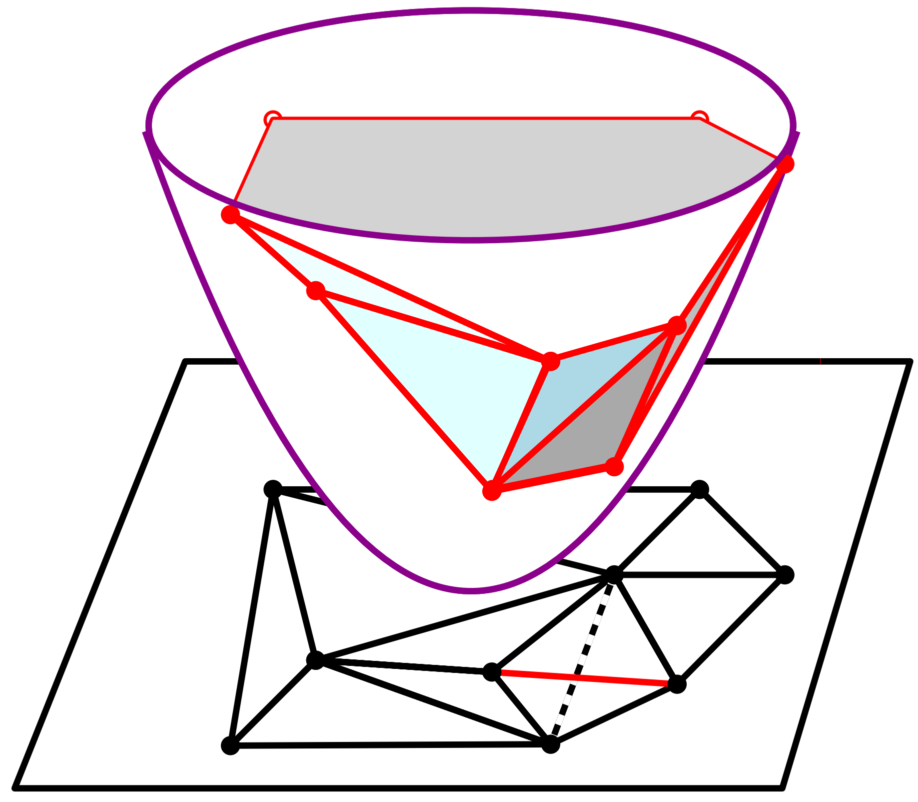


Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



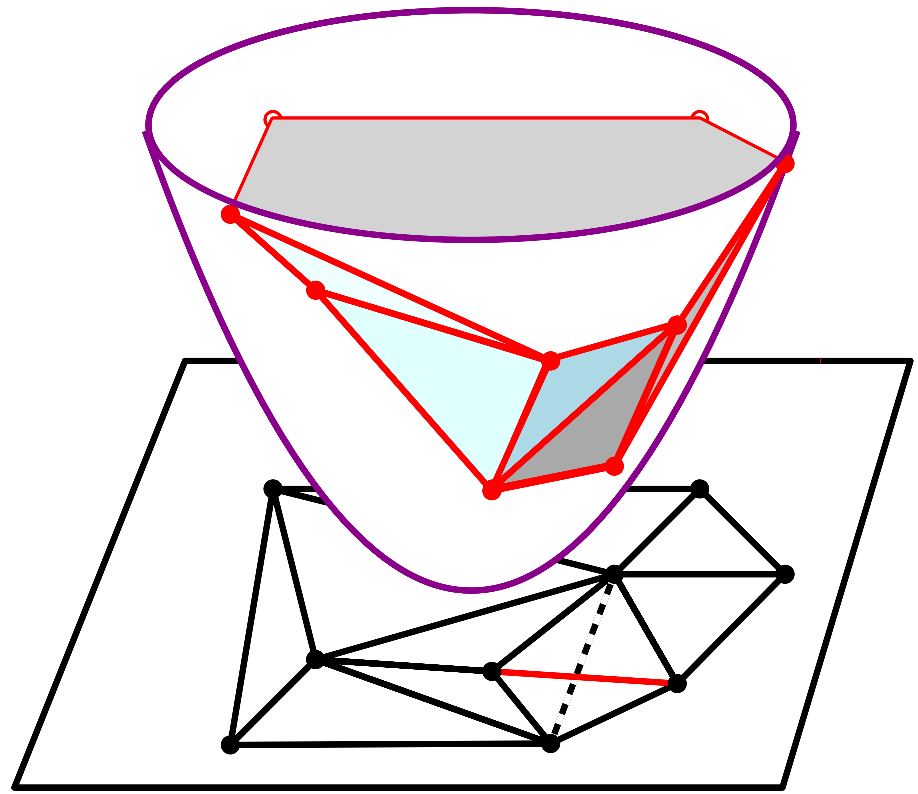
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



An hidden edge cannot be visible again

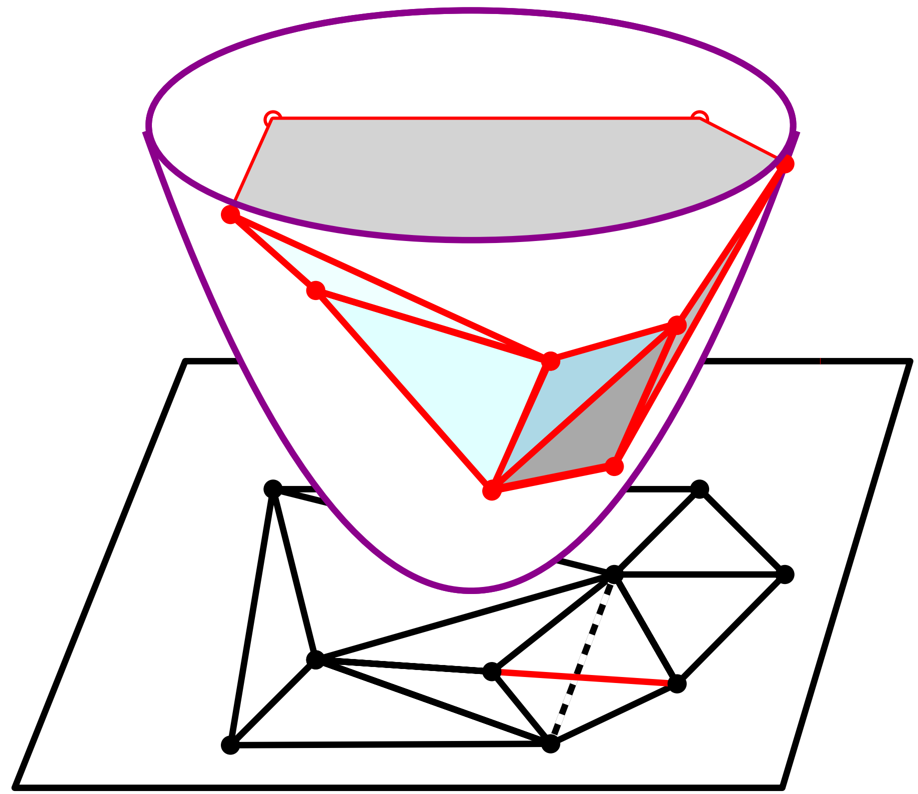
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip

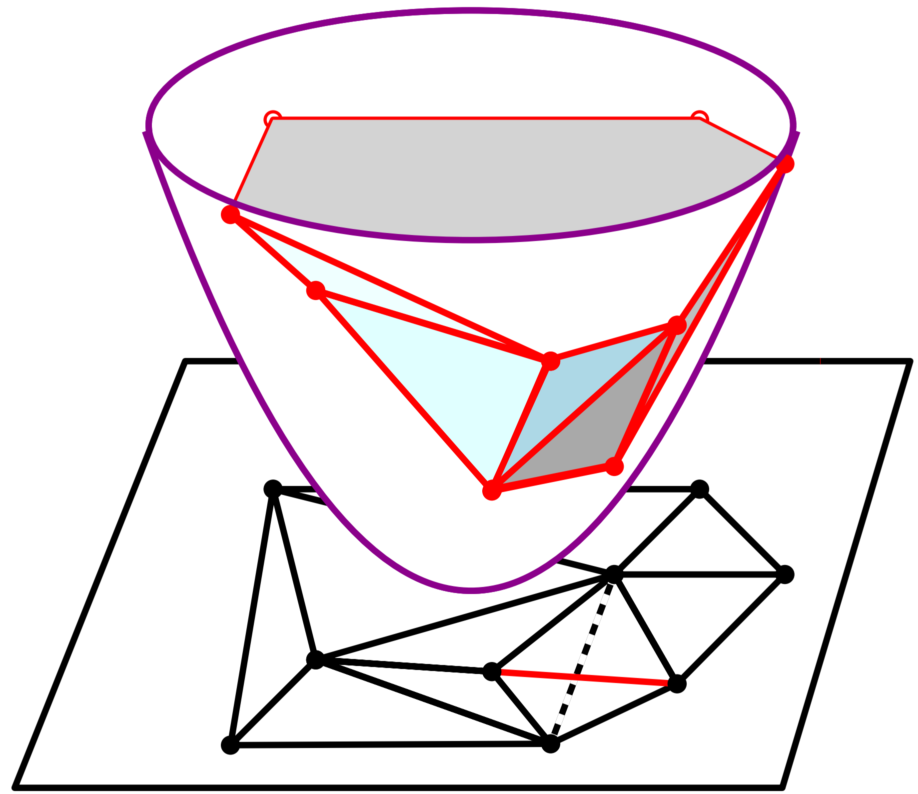


An hidden edge cannot be visible again

Non Delaunay

At most $\frac{n(n-1)}{2}$ edges

Delaunay Triangulation: Diagonal flipping



Complexity ?

Non convex

Flip

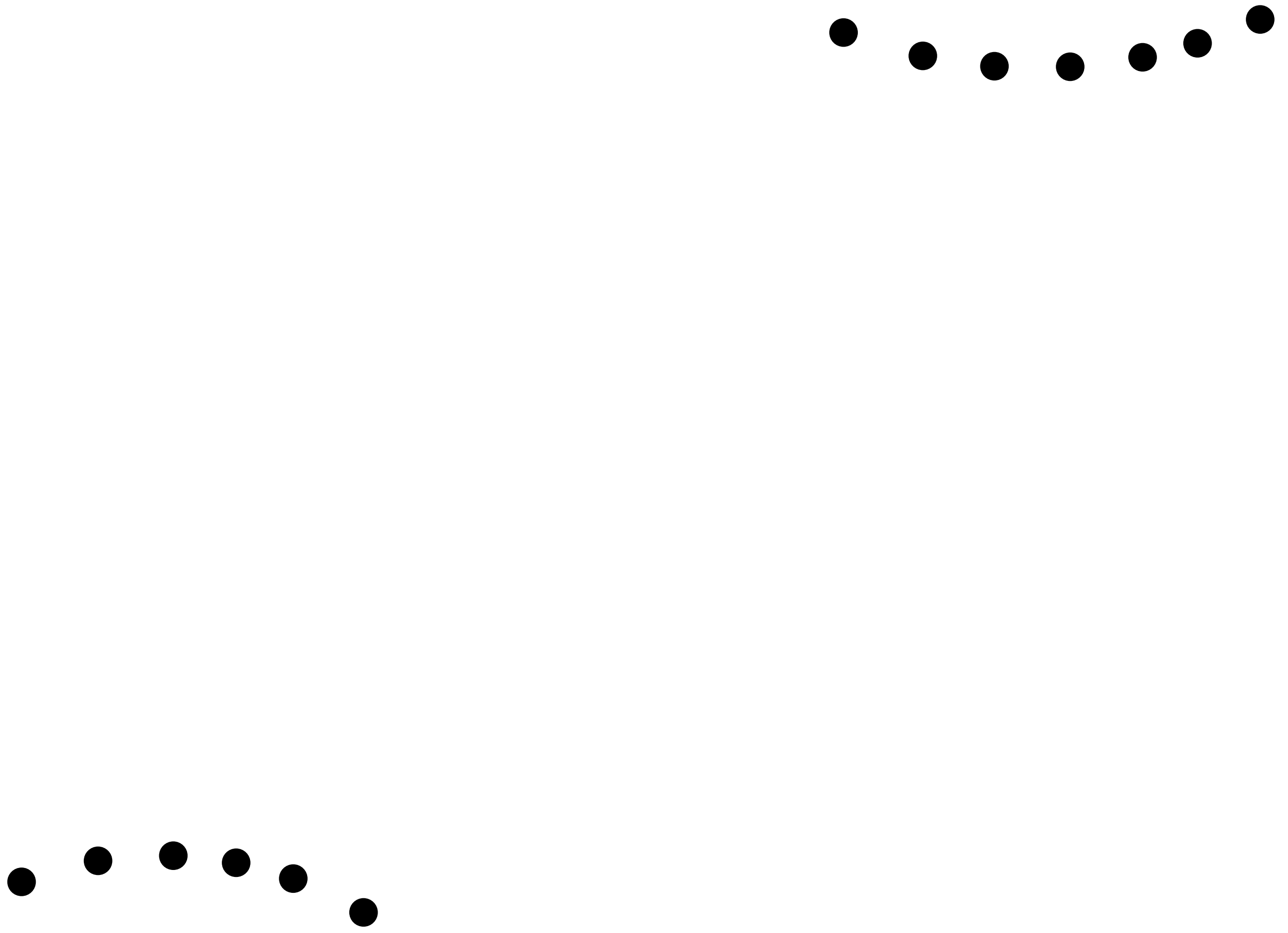
An hidden edge cannot be visible again

At most $\frac{n(n-1)}{2}$ edges

Complexity of diagonal flipping is $O(n^2)$

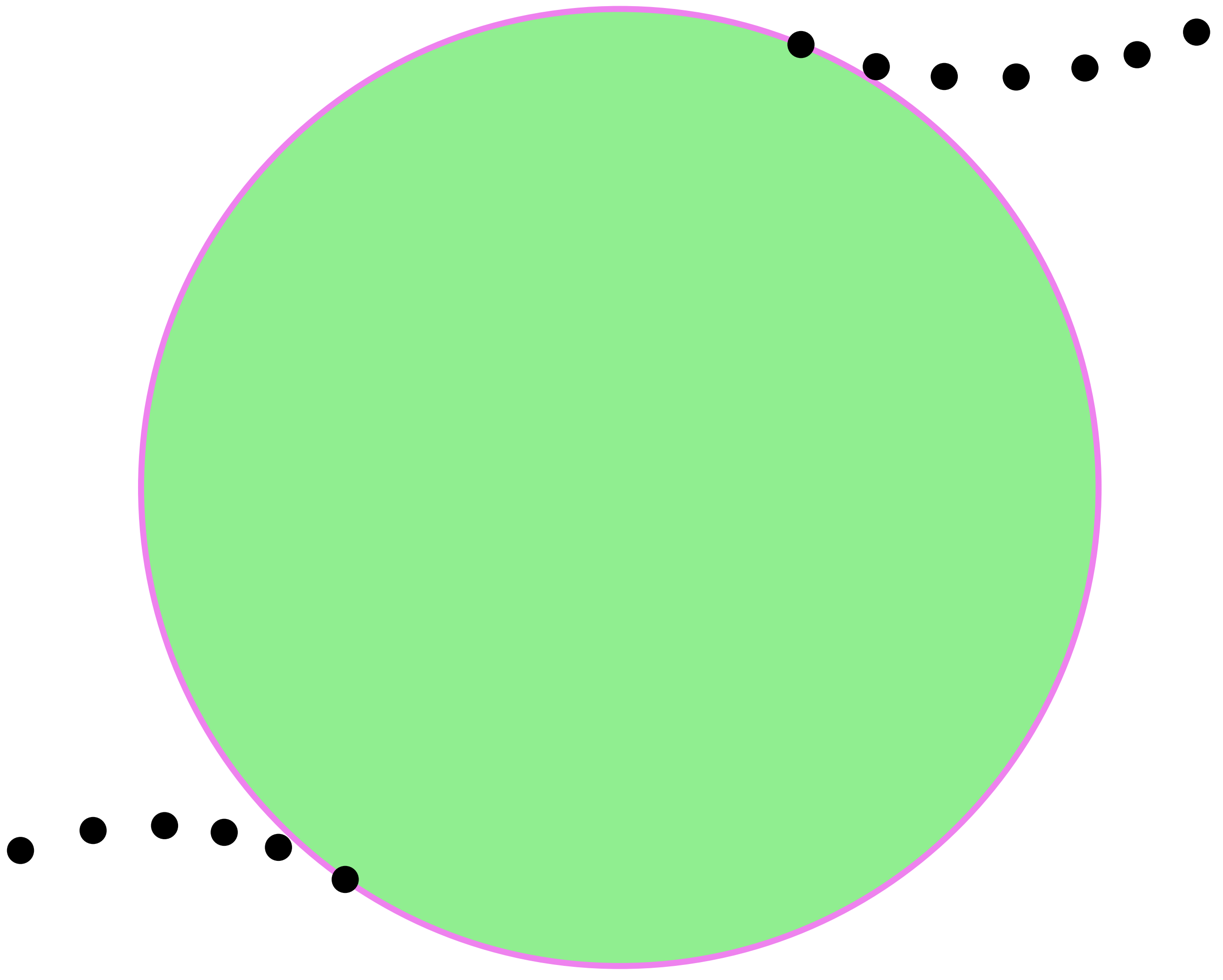
Delaunay Triangulation: Diagonal flipping

Complexity ?



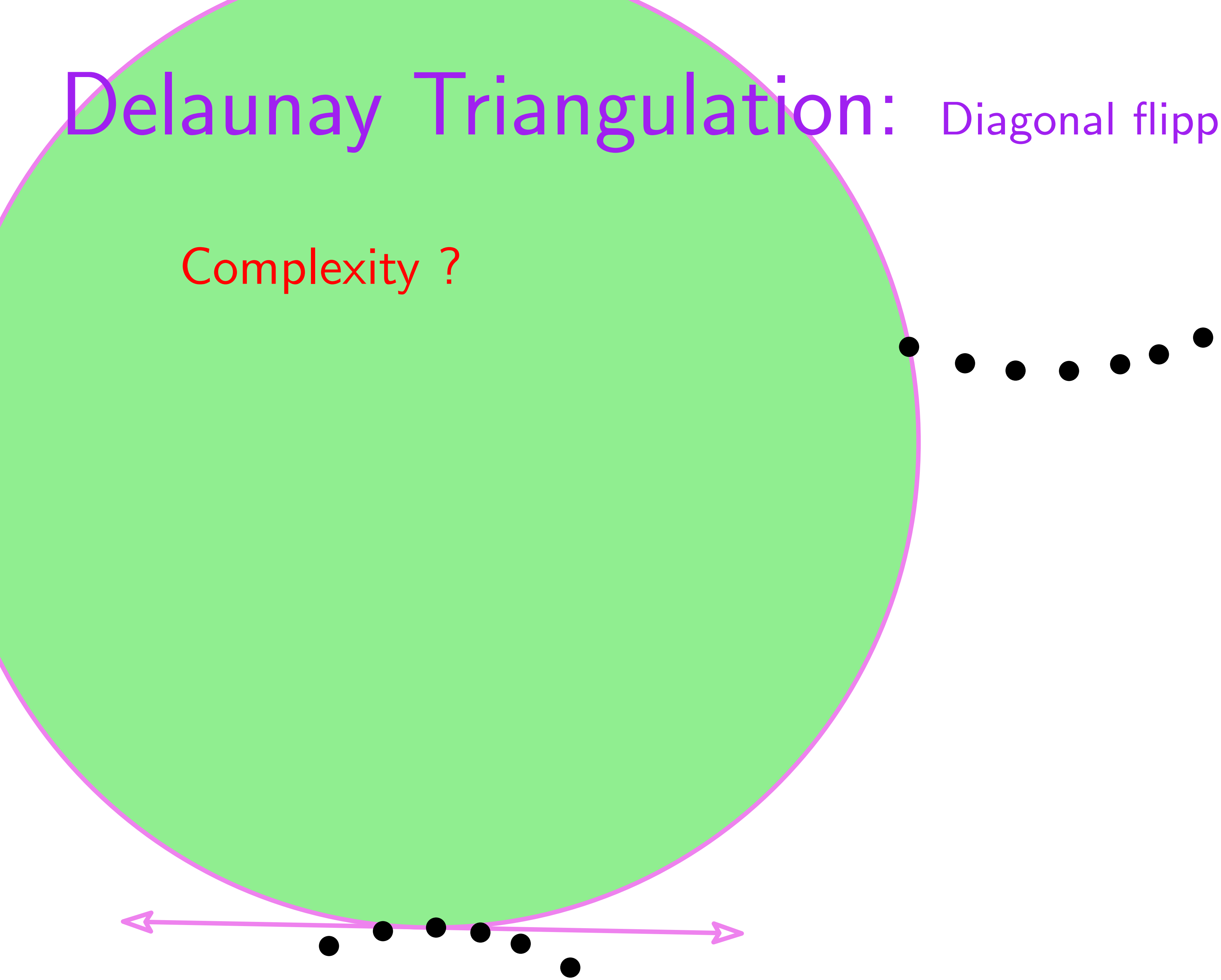
Delaunay Triangulation: Diagonal flipping

Complexity ?



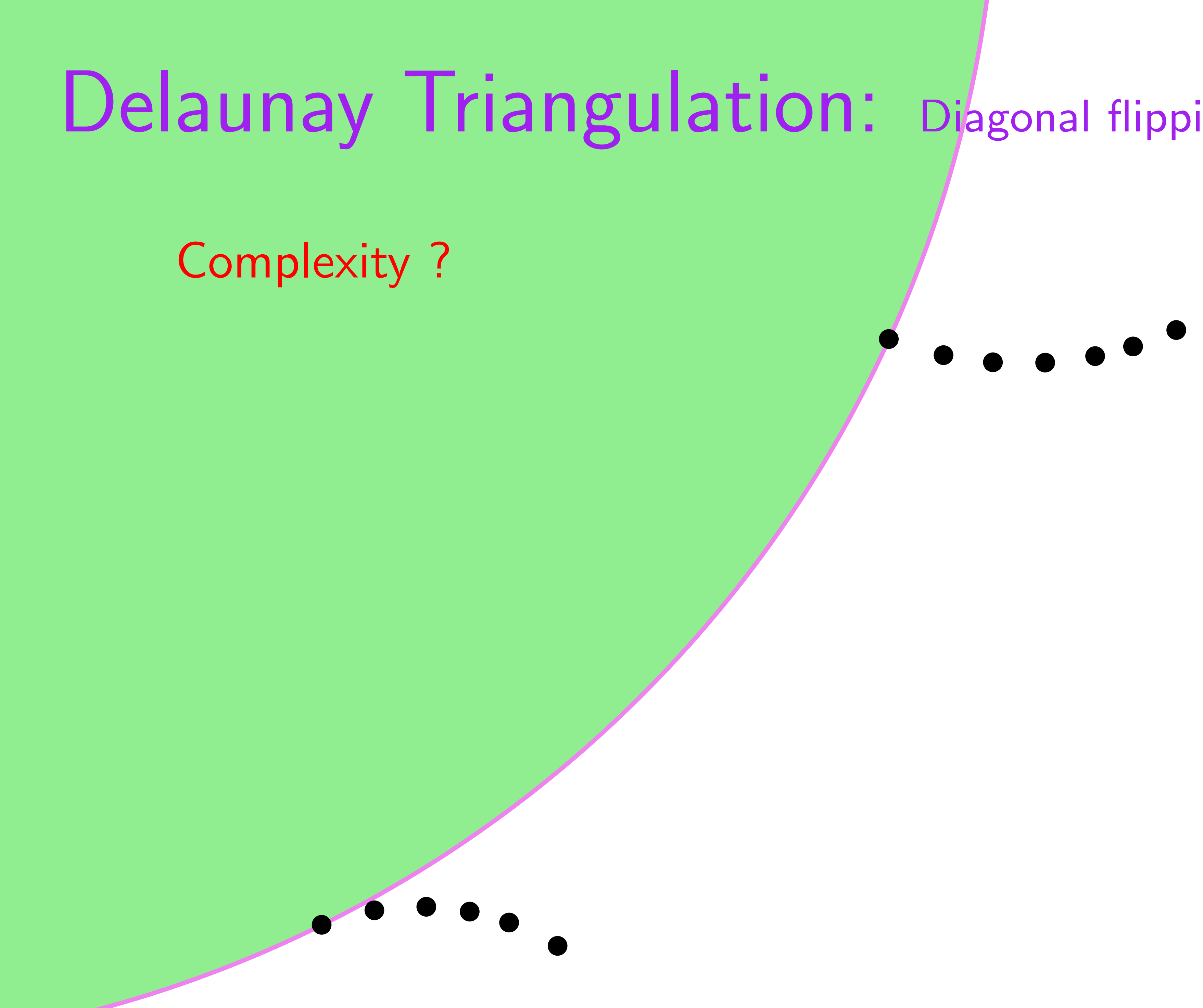
Delaunay Triangulation: Diagonal flipping

Complexity ?



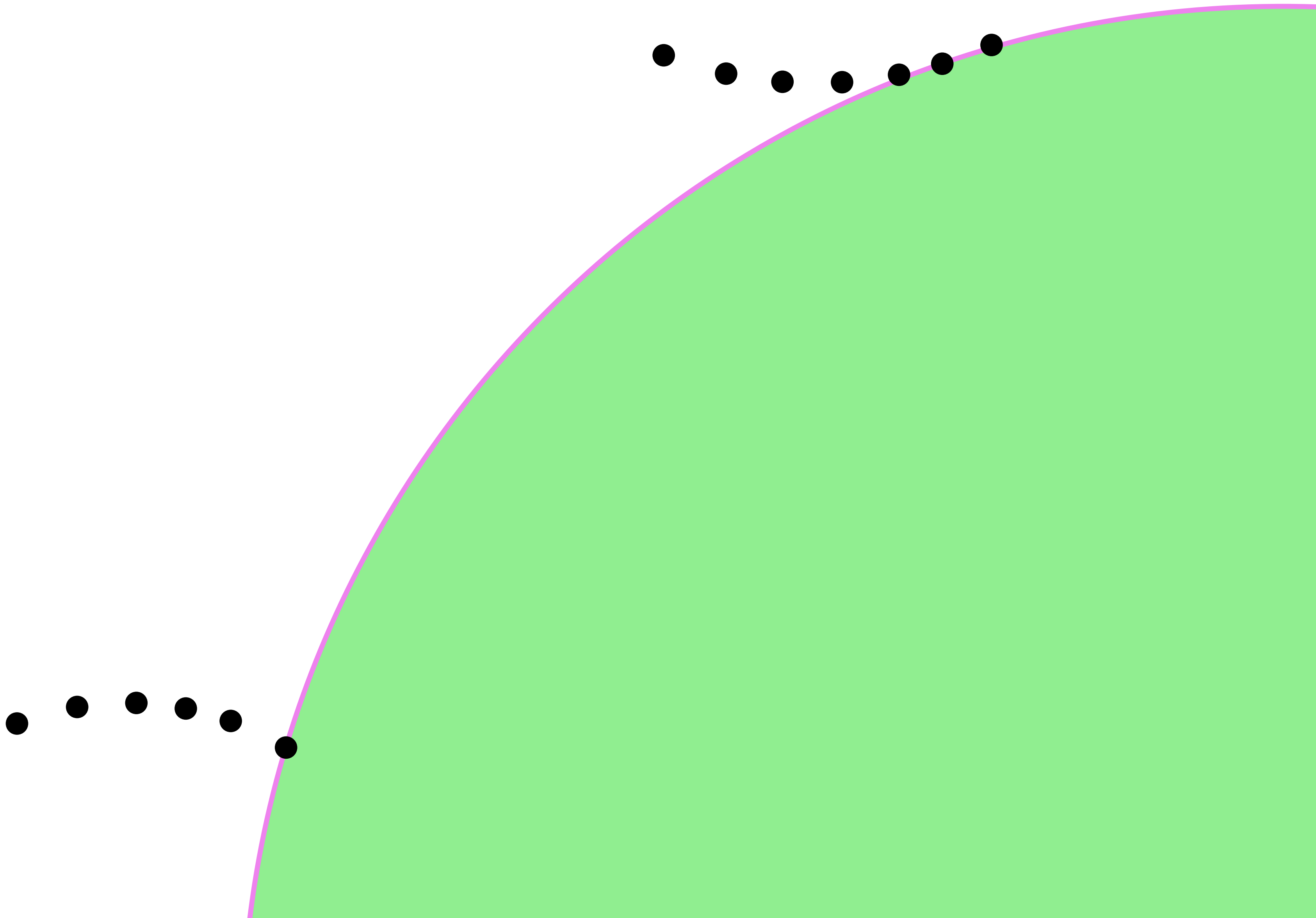
Delaunay Triangulation: Diagonal flipping

Complexity ?



Delaunay Triangulation: Diagonal flipping

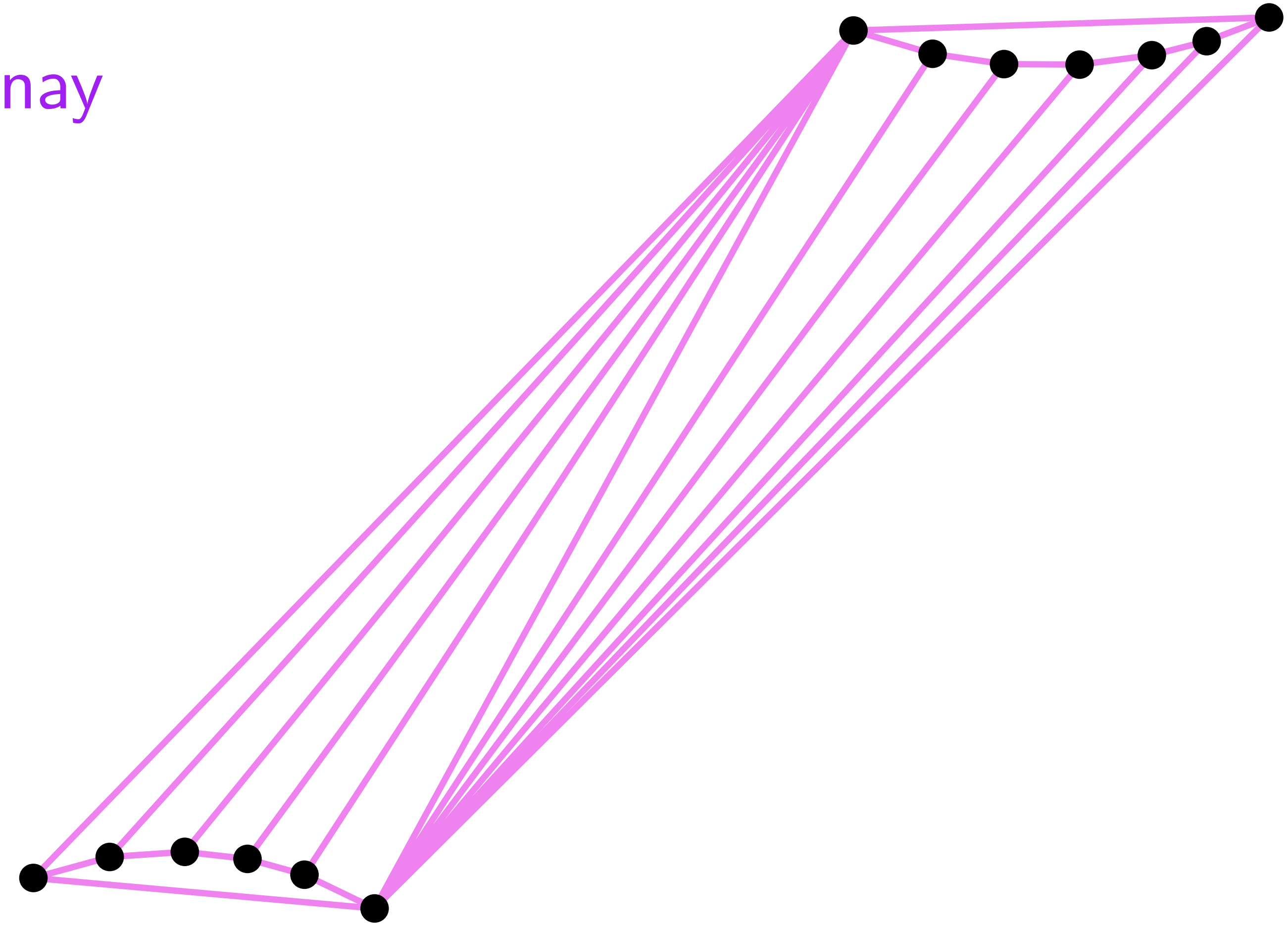
Complexity ?



Delaunay Triangulation: Diagonal flipping

Complexity ?

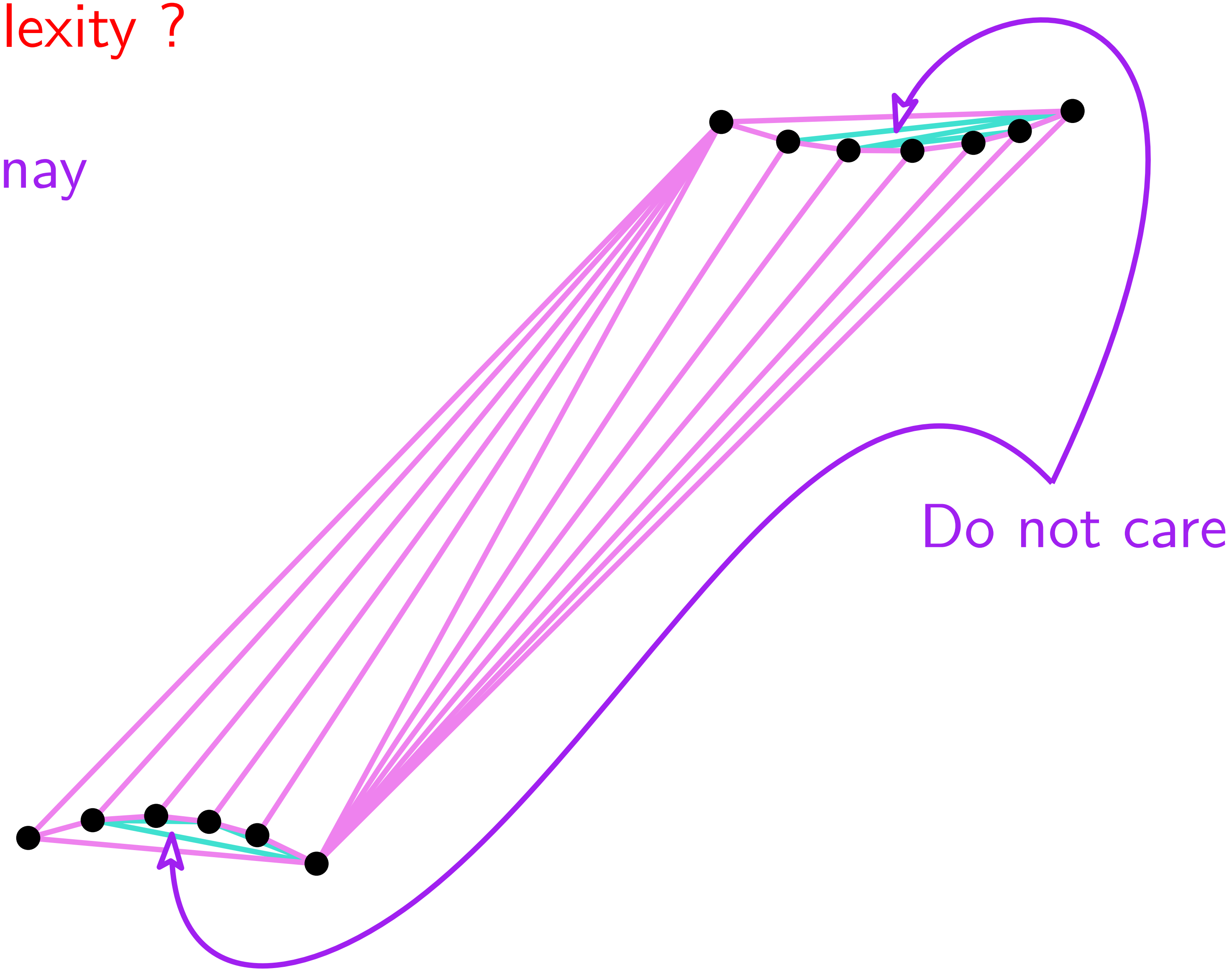
Delaunay



Delaunay Triangulation: Diagonal flipping

Complexity ?

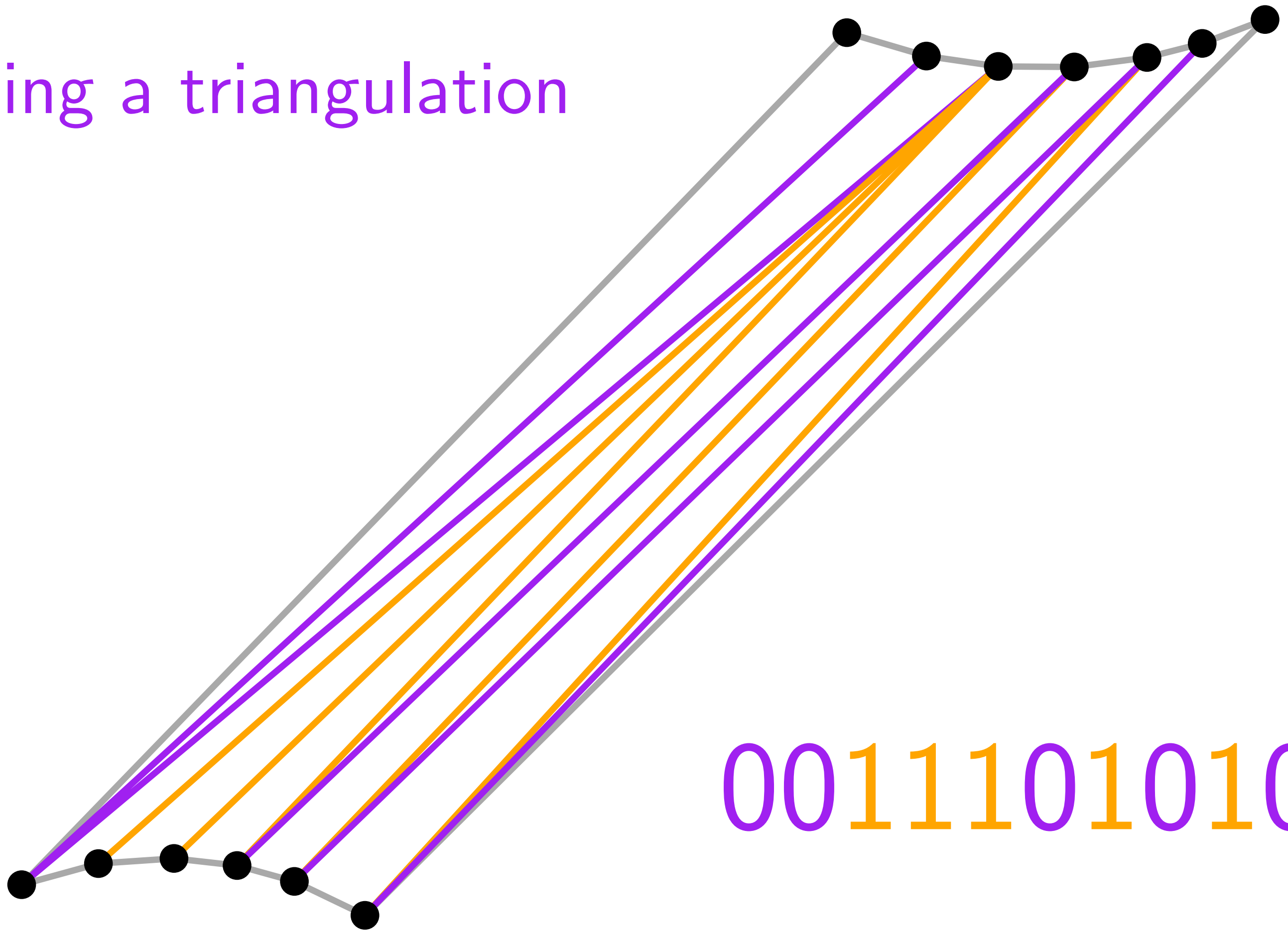
Delaunay



Delaunay Triangulation: Diagonal flipping

Complexity ?

Encoding a triangulation

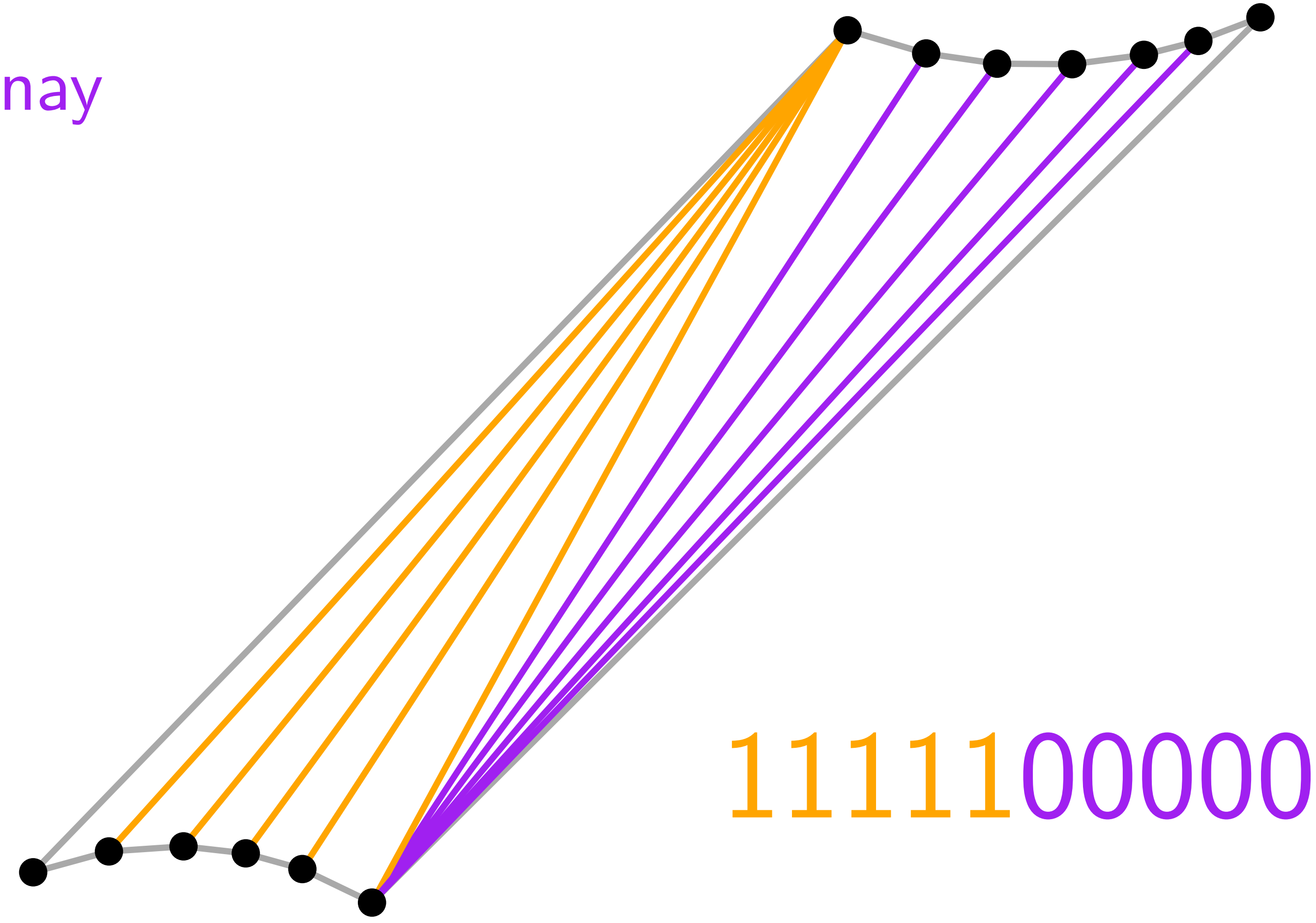


0011101010

Delaunay Triangulation: Diagonal flipping

Complexity ?

Delaunay

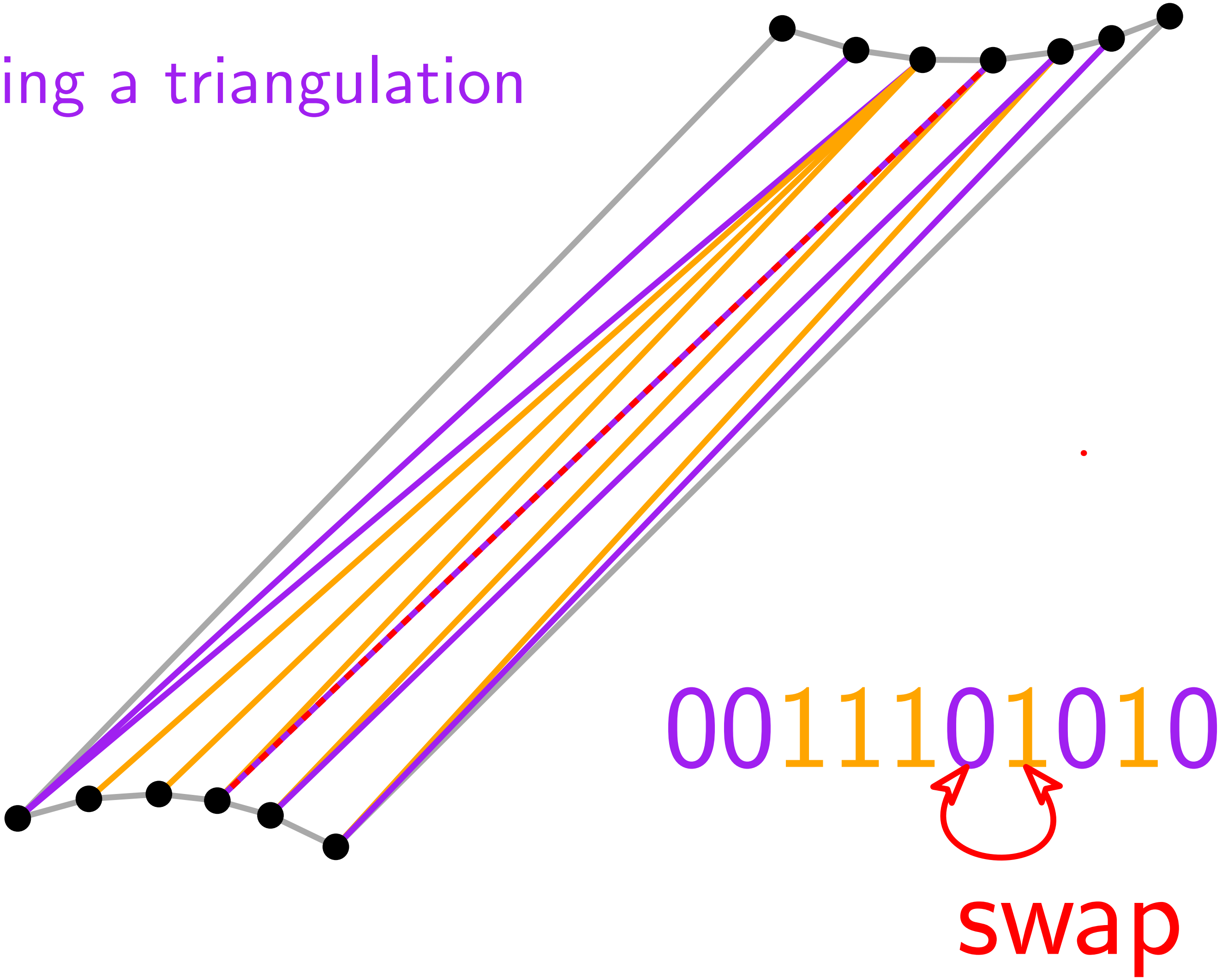


Delaunay Triangulation: Diagonal flipping

Complexity ?

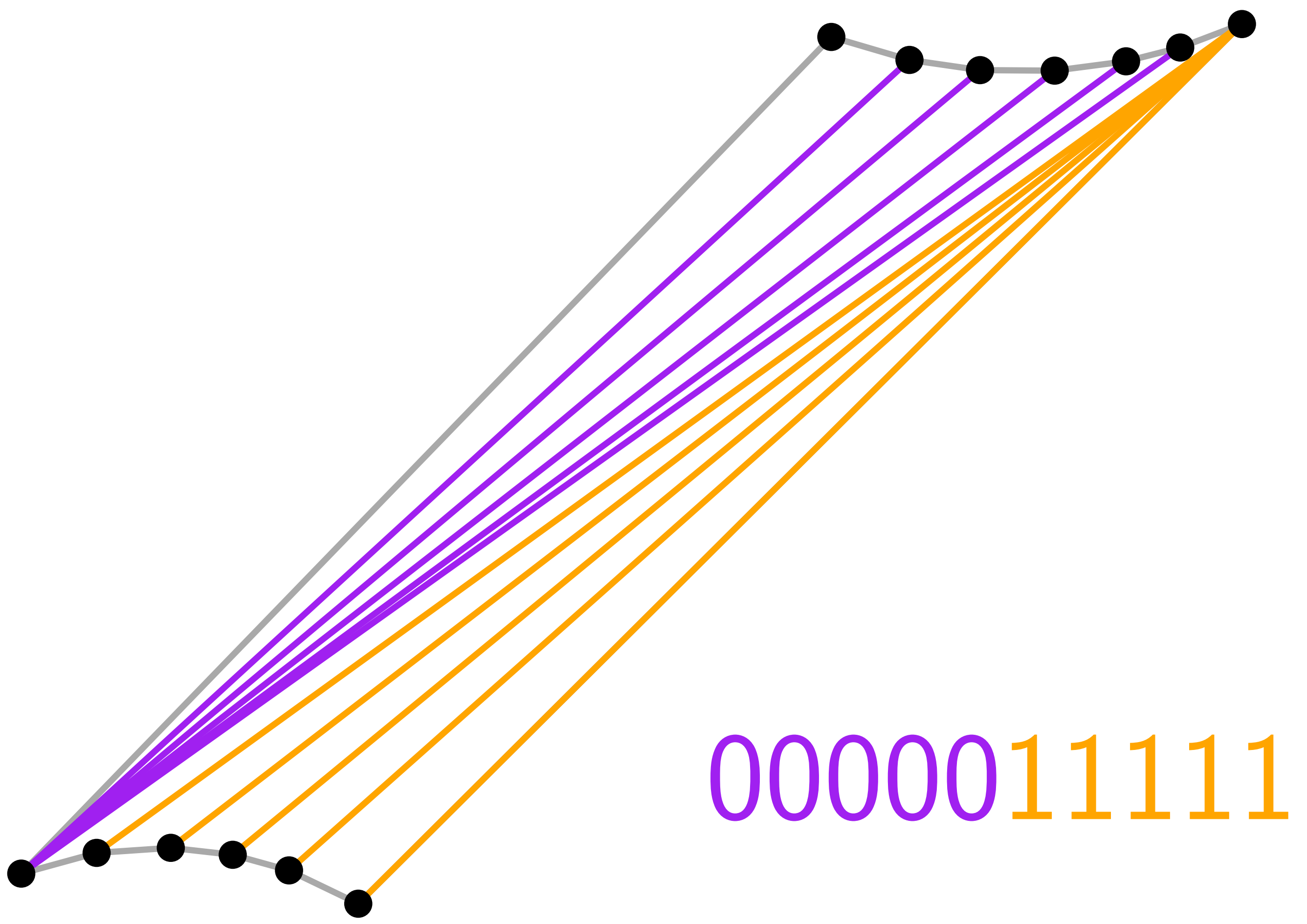
Encoding a triangulation

Flip



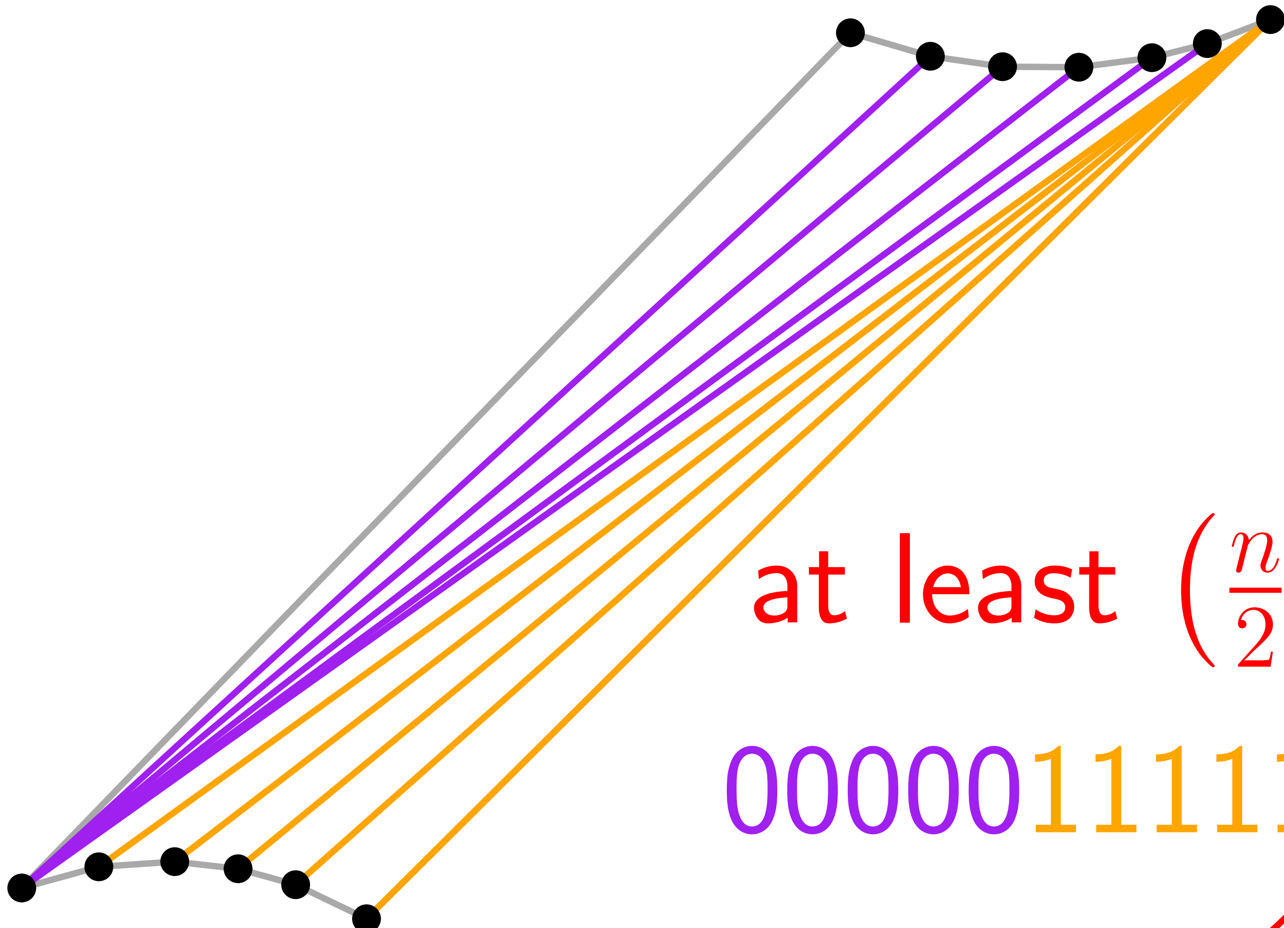
Delaunay Triangulation: Diagonal flipping

Complexity ?



Delaunay Triangulation: Diagonal flipping

Complexity ?



at least $\left(\frac{n}{2}\right)^2$ flips

0000011111

Delaunay 1111100000

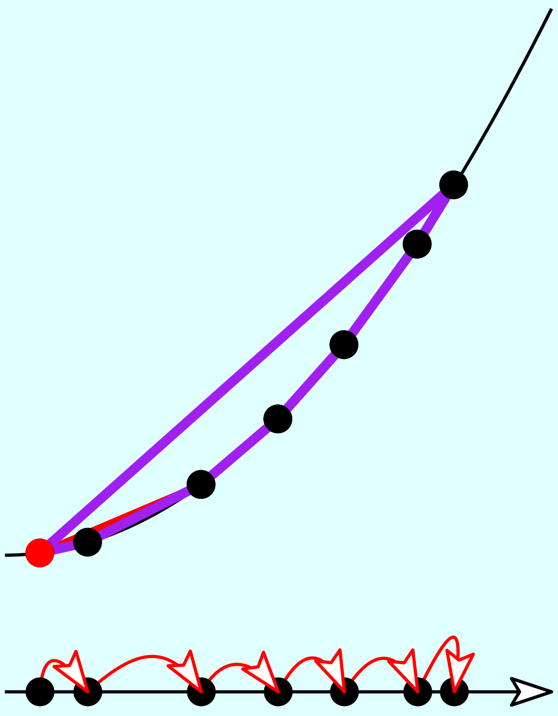
Borne inférieure de complexité

Delaunay Triangulation: lower bound

Convex hull

Lower bound

A stupid algorithm for sorting numbers



$O(n)$ project on parabola

$f(n)$ compute convex hull

$O(n)$ find lowest point

$O(n)$ enumerate x coordinates in ccw CH order

Lower bound on sorting

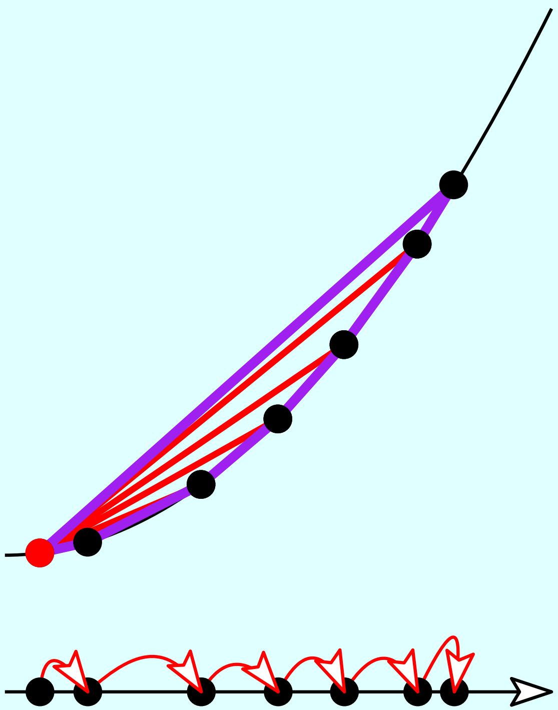
$\Longrightarrow f(n) + O(n) \geq \Omega(n \log n)$

Delaunay Triangulation: lower bound

Convex hull

Lower bound

A stupid algorithm for sorting numbers



$O(n)$ project on parabola

$f(n)$ compute ~~convex hull~~ Delaunay

$O(n)$ find lowest point

$O(n)$ enumerate x coordinates in ccw CH order

Lower bound on sorting

$\Longrightarrow f(n) + O(n) \geq \Omega(n \log n)$

Point location in Delaunay

Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

$$x^2 + y^2 - 2ax - 2by + c$$

Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

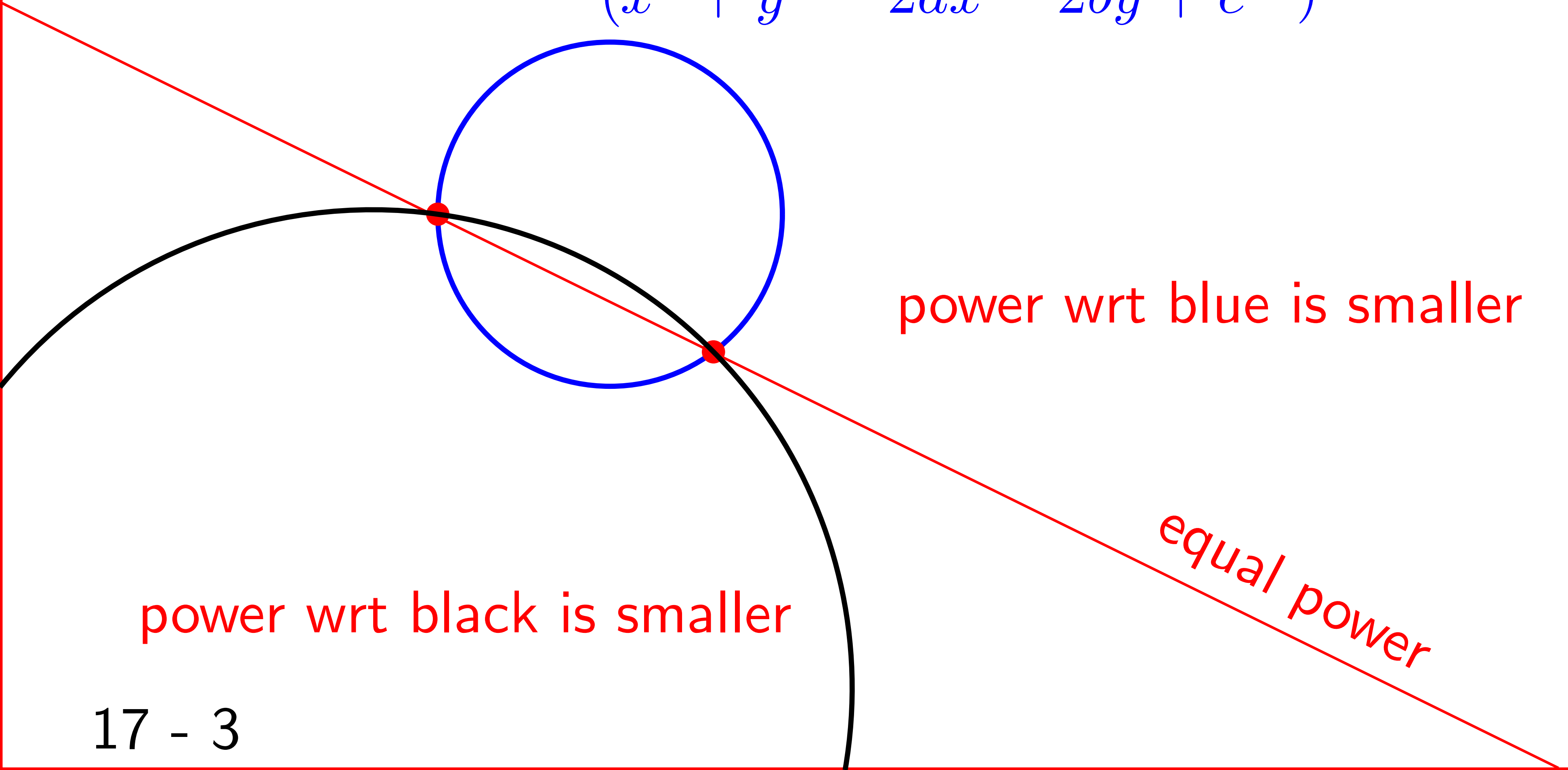
$$x^2 + y^2 - 2ax - 2by + c$$

$= 0$	on the circle
< 0	inside the circle
> 0	outside the circle

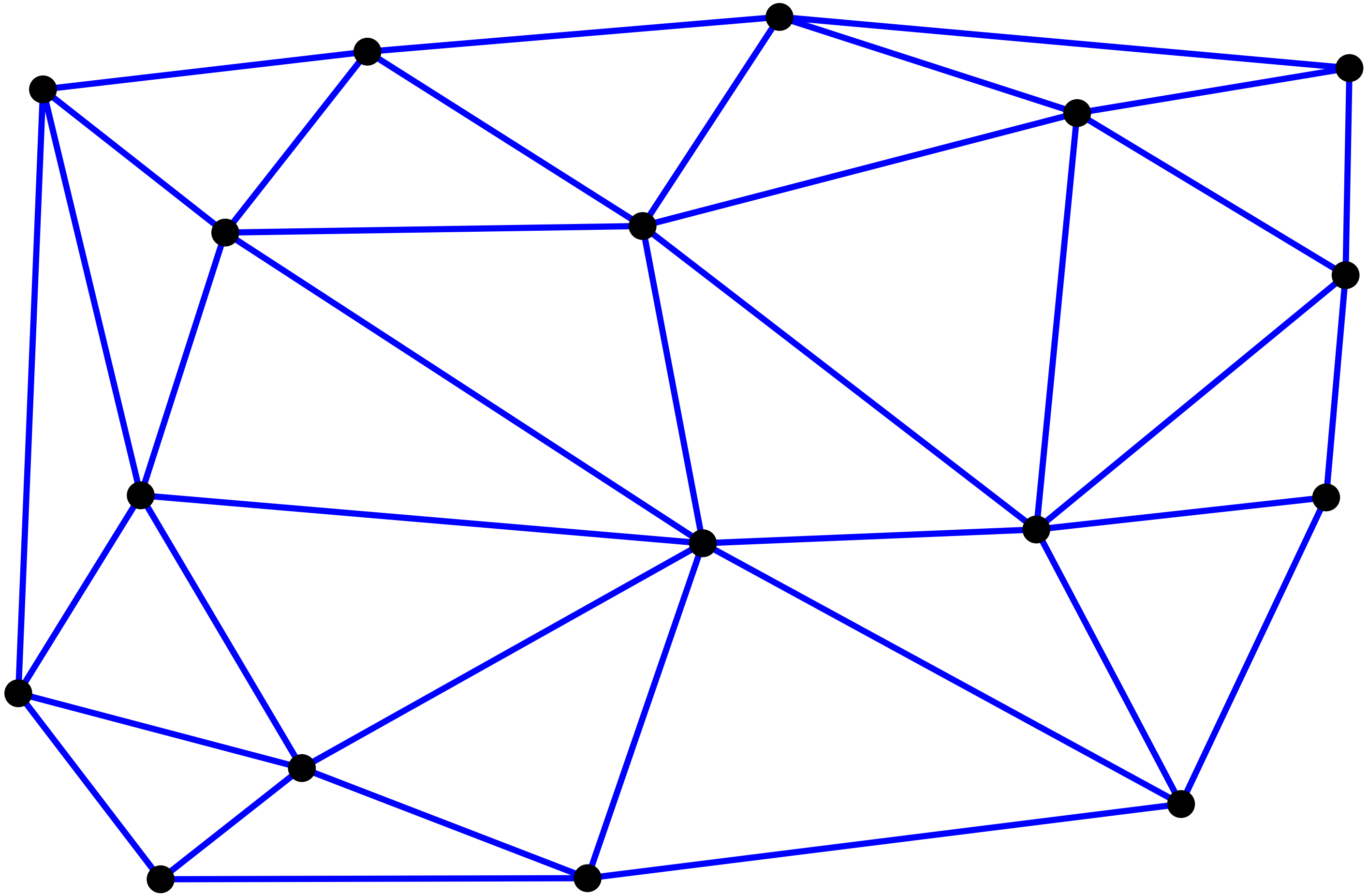
Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

$$(x^2 + y^2 - 2a'x - 2b'y + c')$$
$$- \ (x^2 + y^2 - 2ax - 2by + c \quad)$$

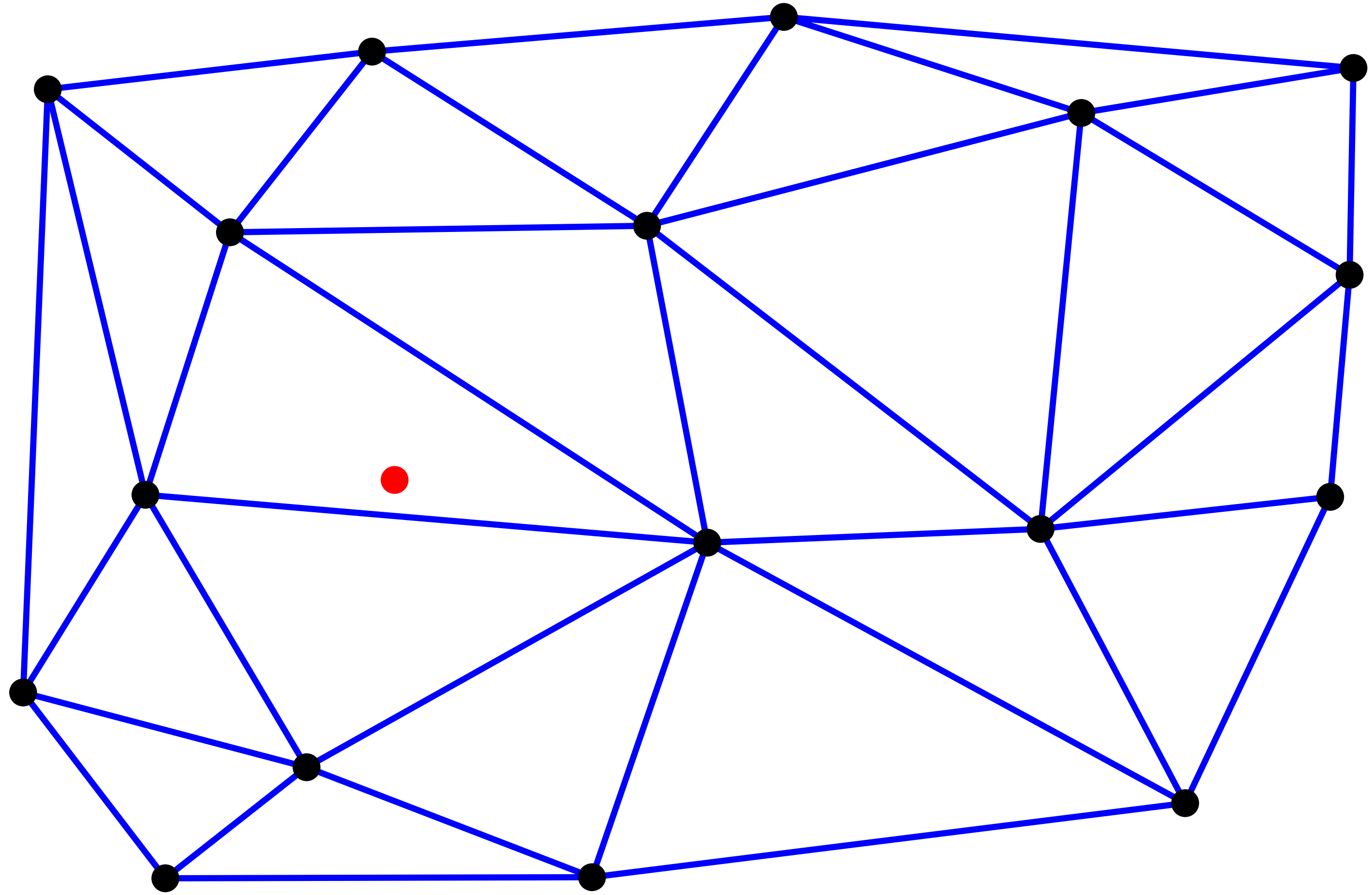


Delaunay Triangulation: incremental algorithm



Delaunay Triangulation: incremental algorithm

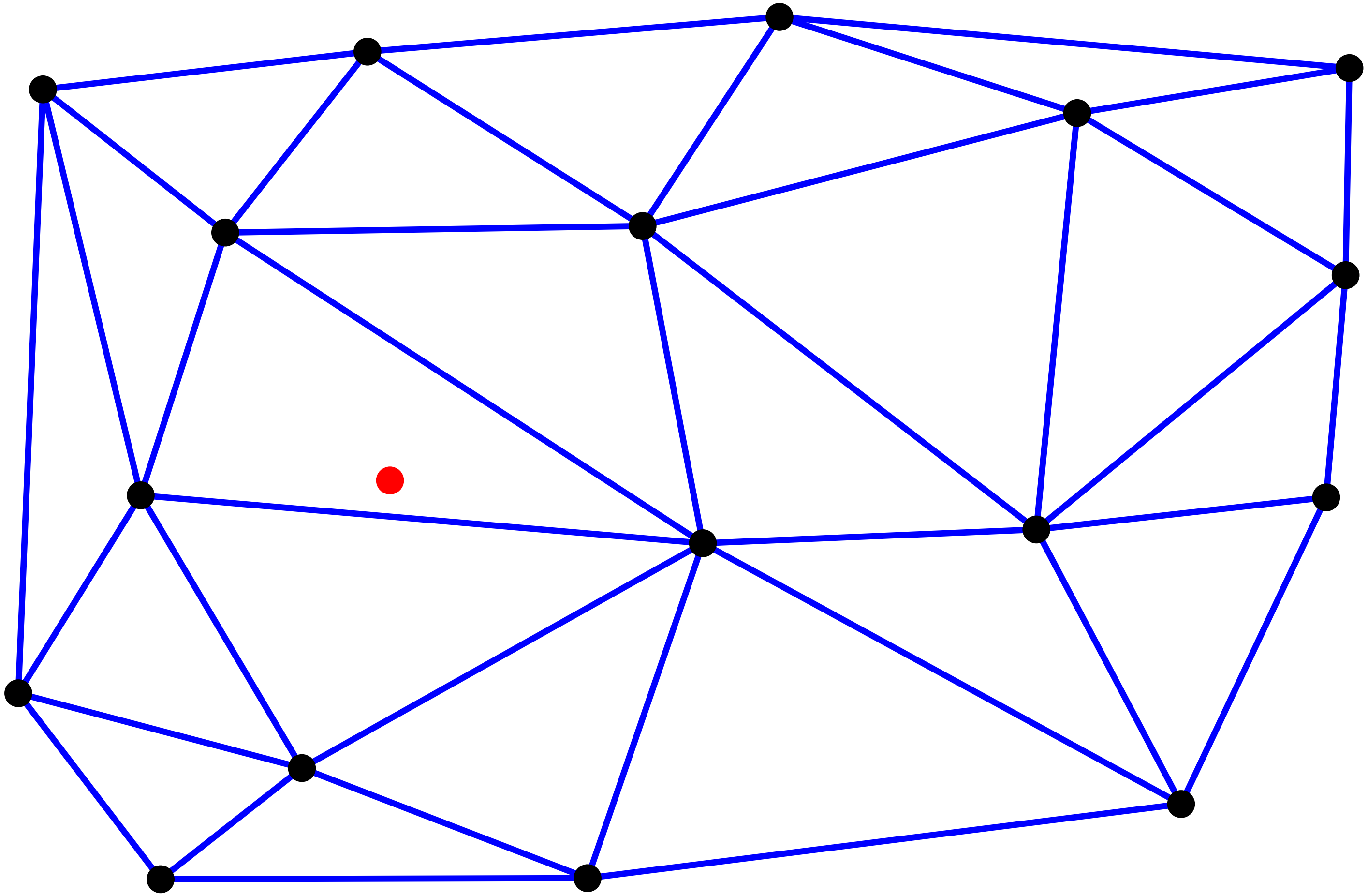
New point



Delaunay Triangulation: incremental algorithm

New point

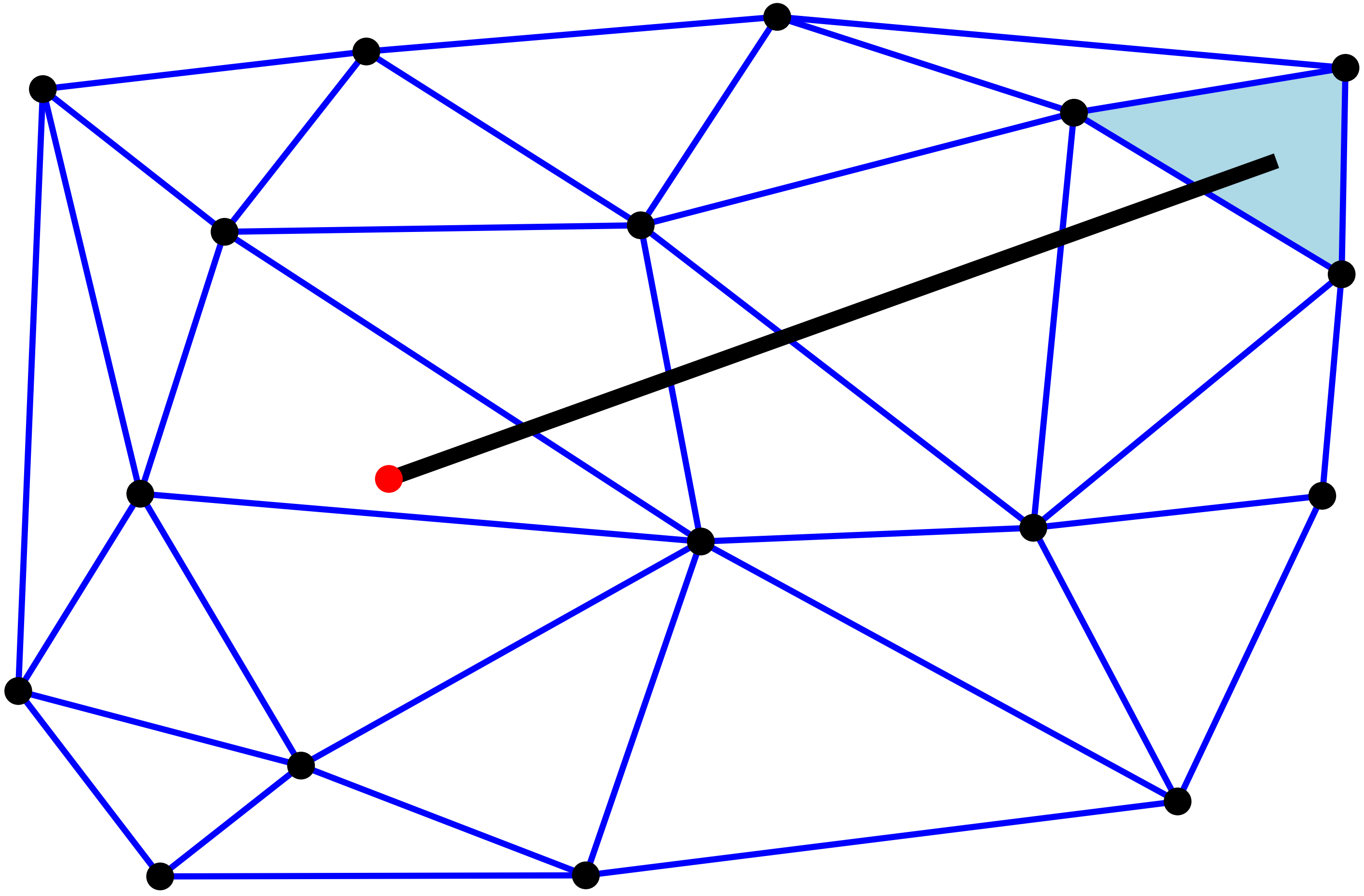
Locate



Delaunay Triangulation: incremental algorithm

New point

Locate

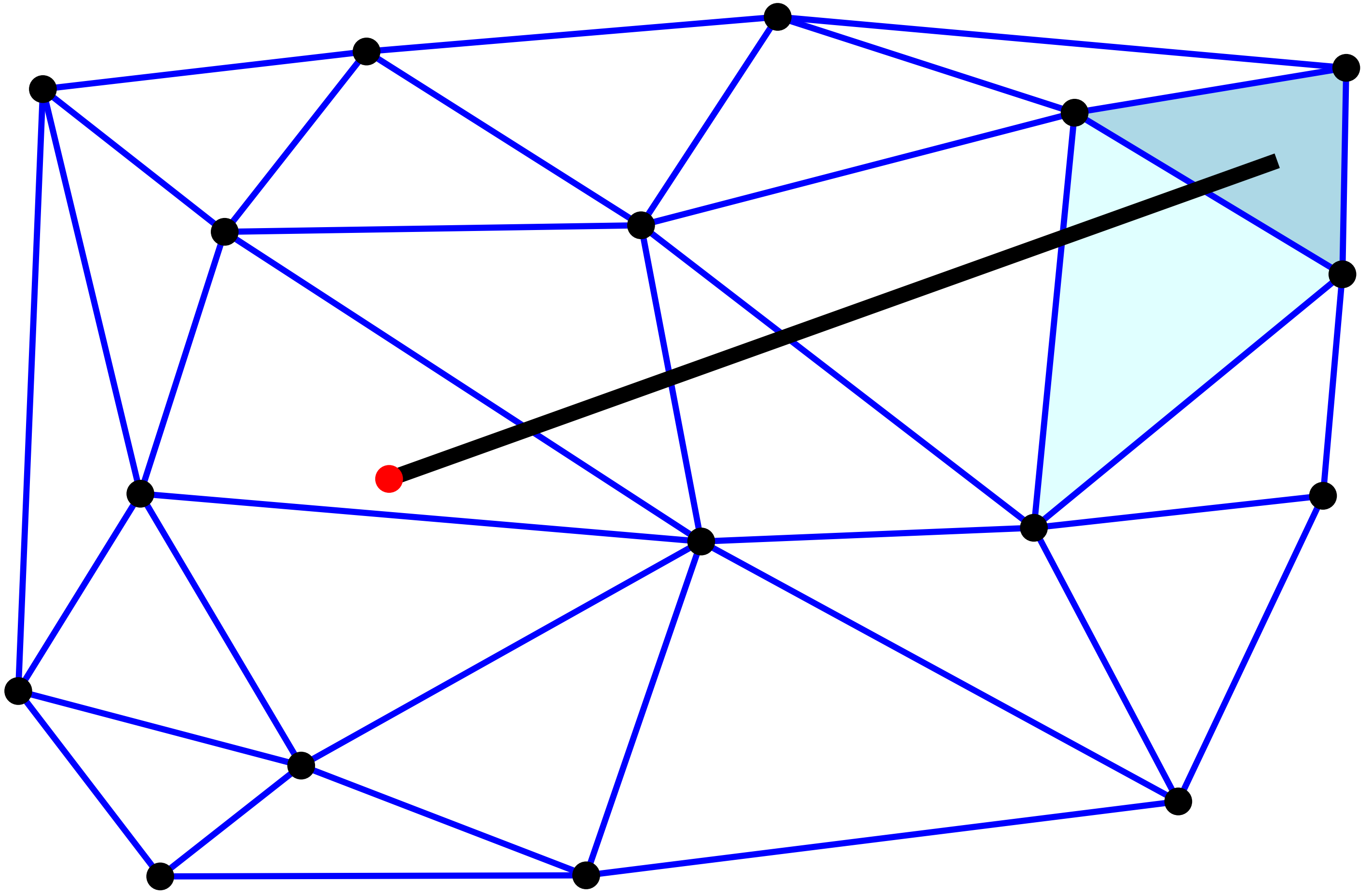


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

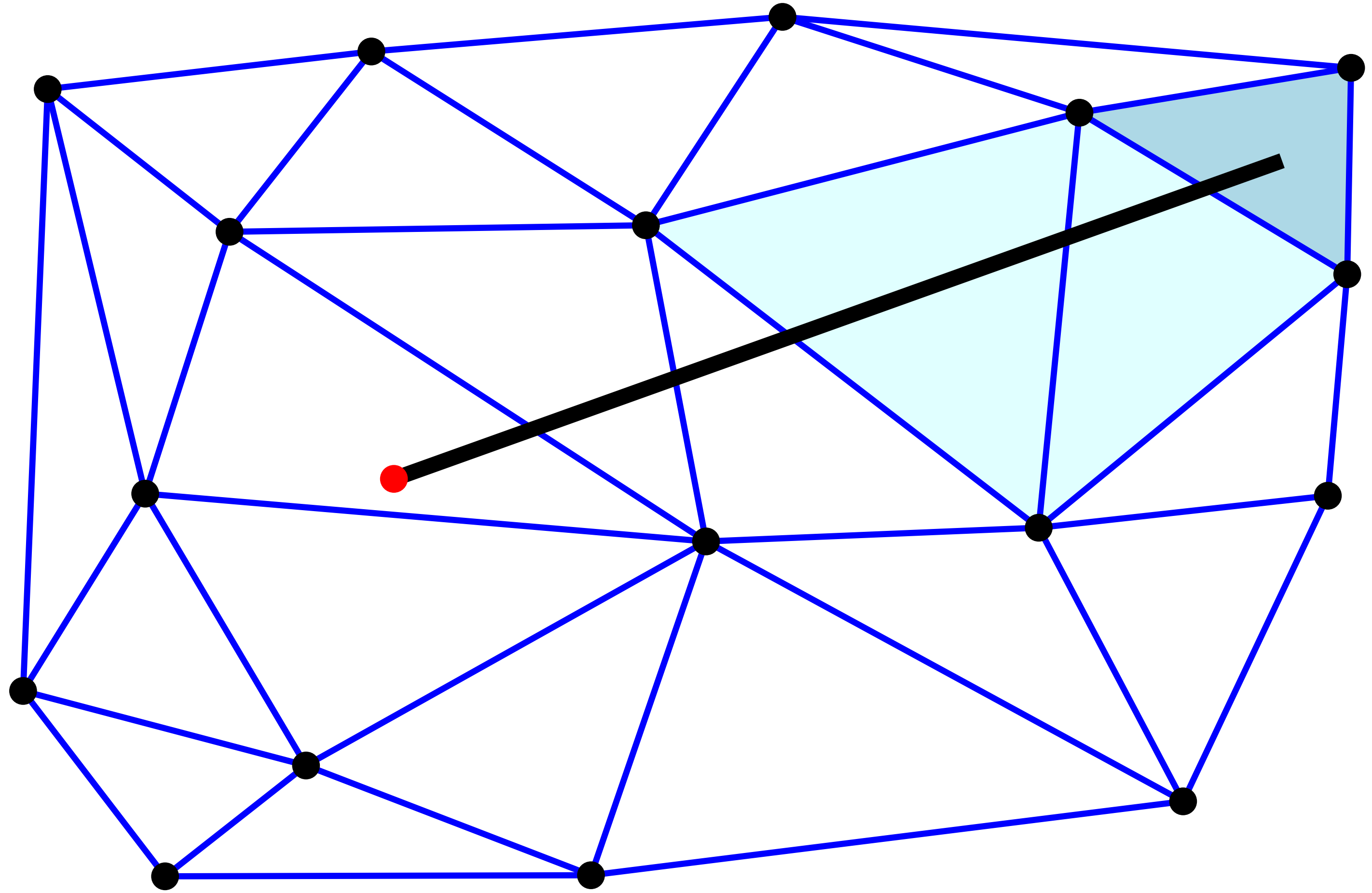


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

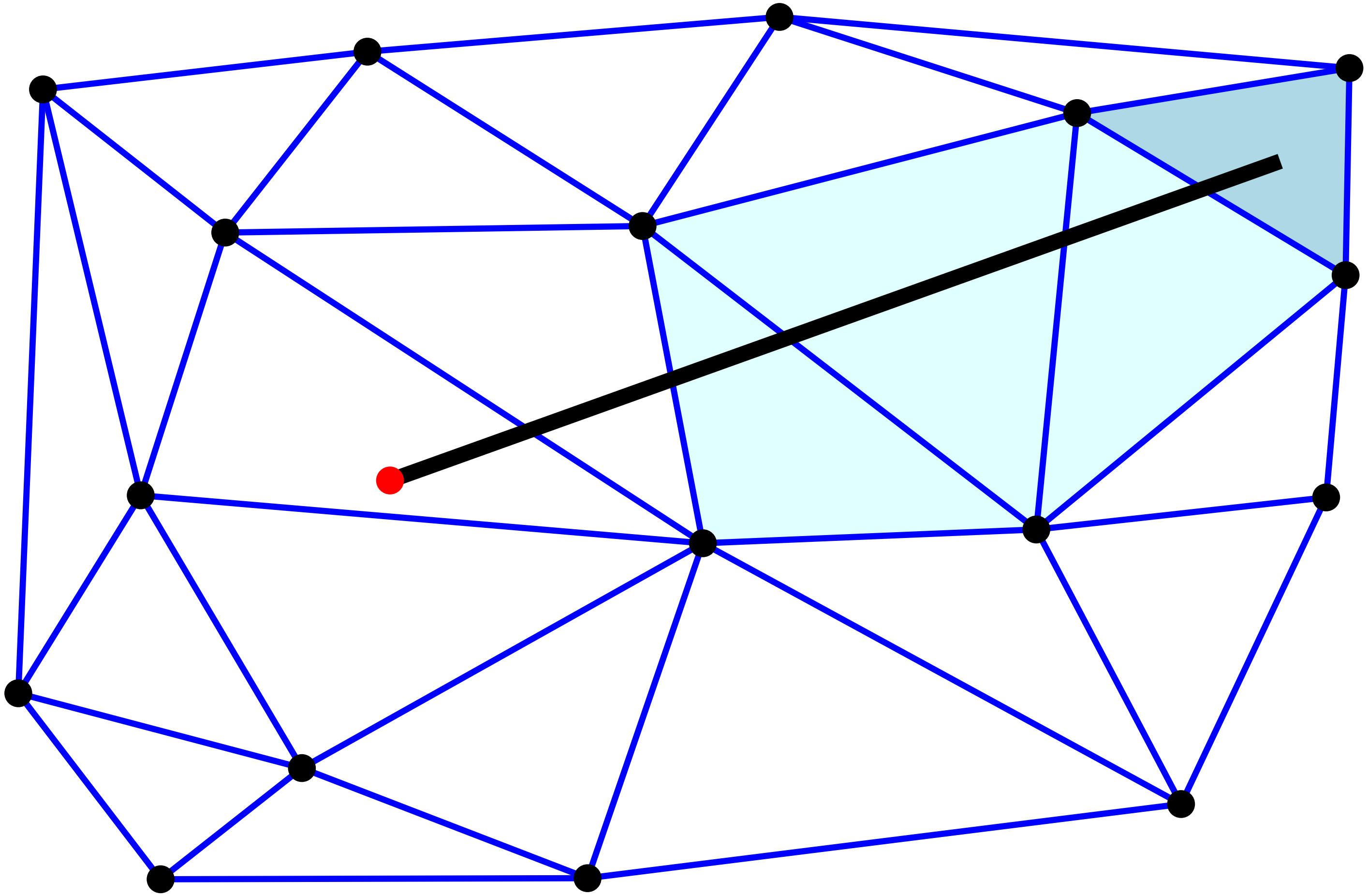


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

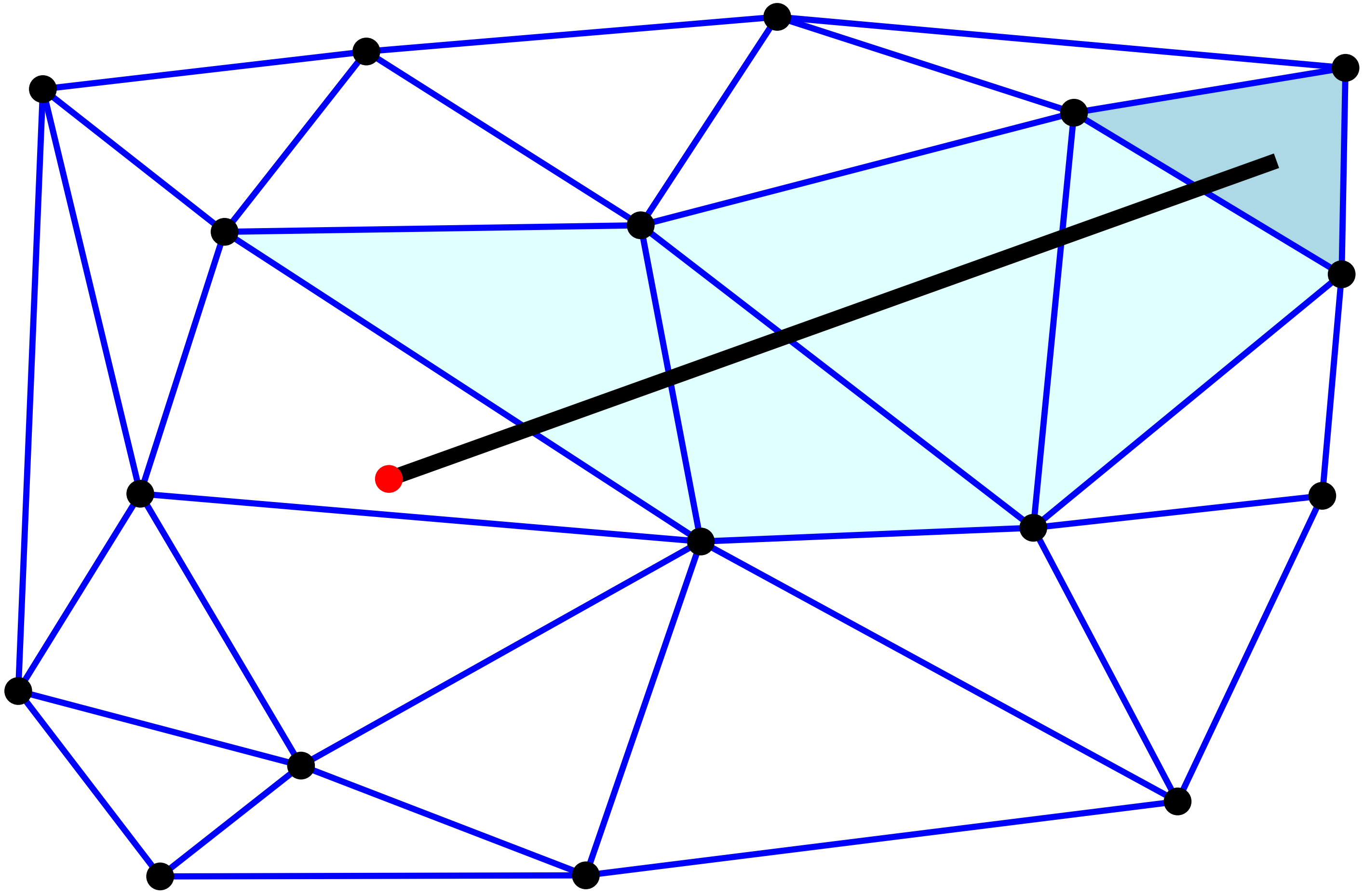


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

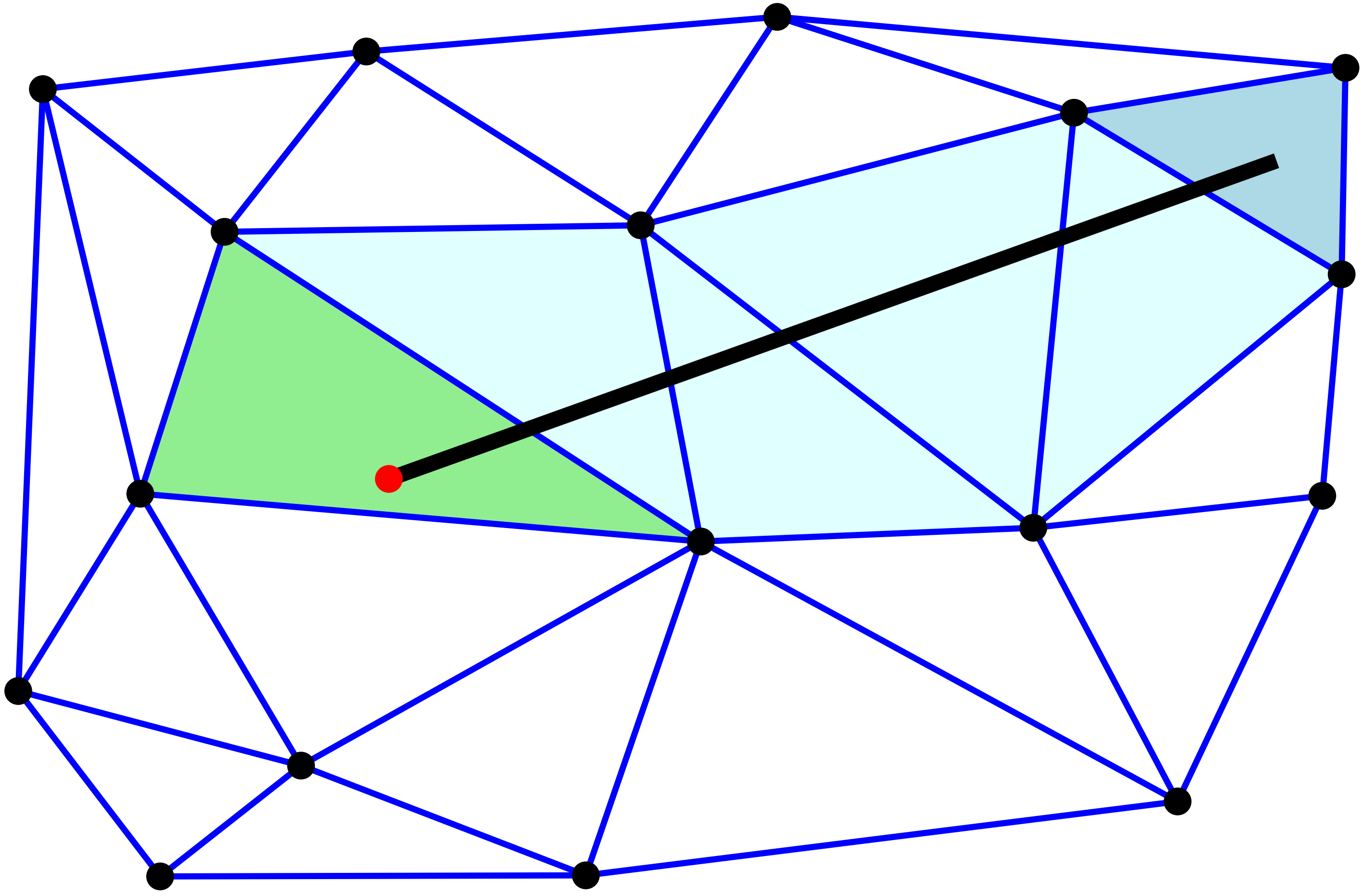


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

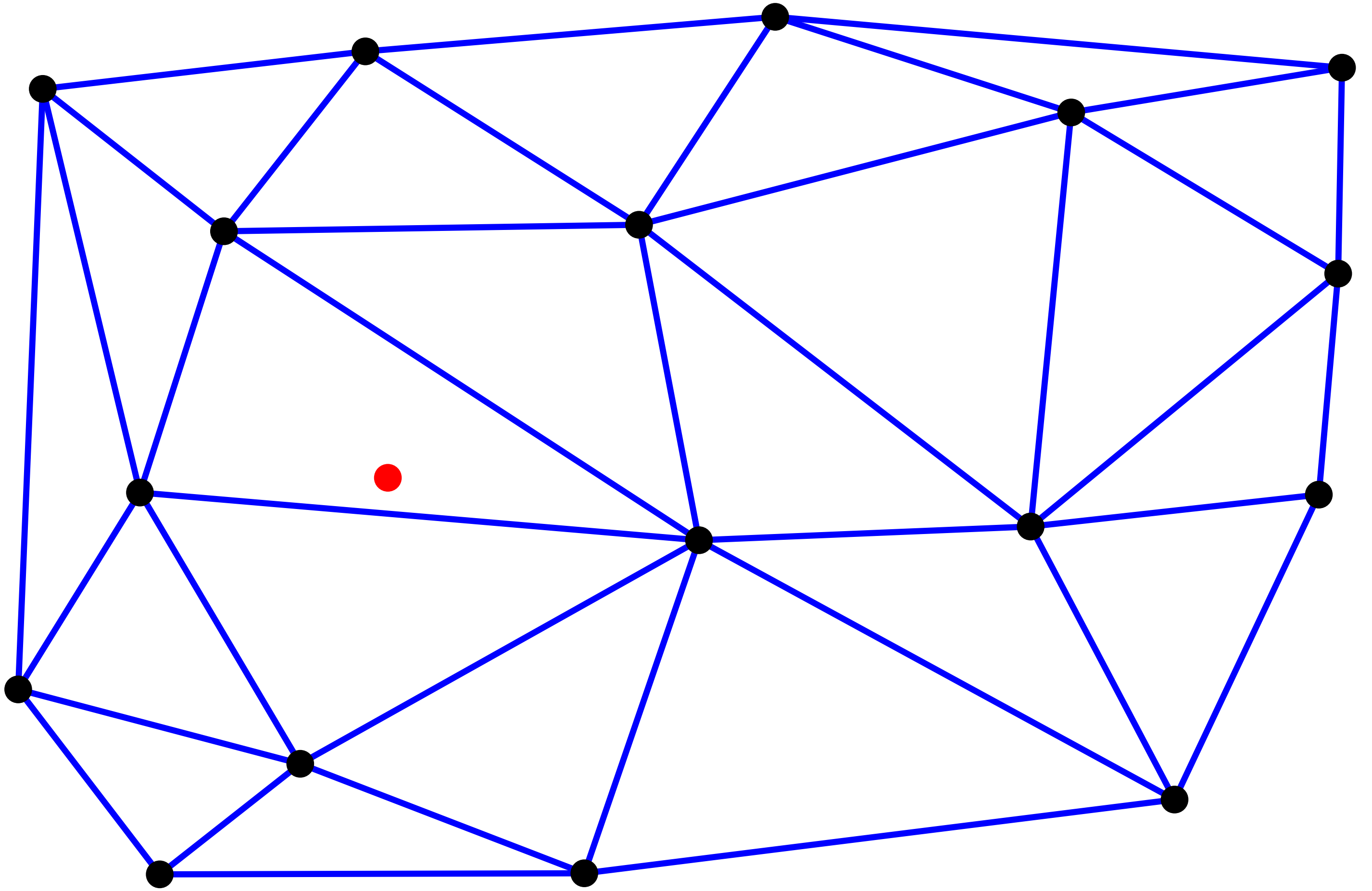


e.g.: straight walk

Delaunay Triangulation: incremental algorithm

New point

Locate

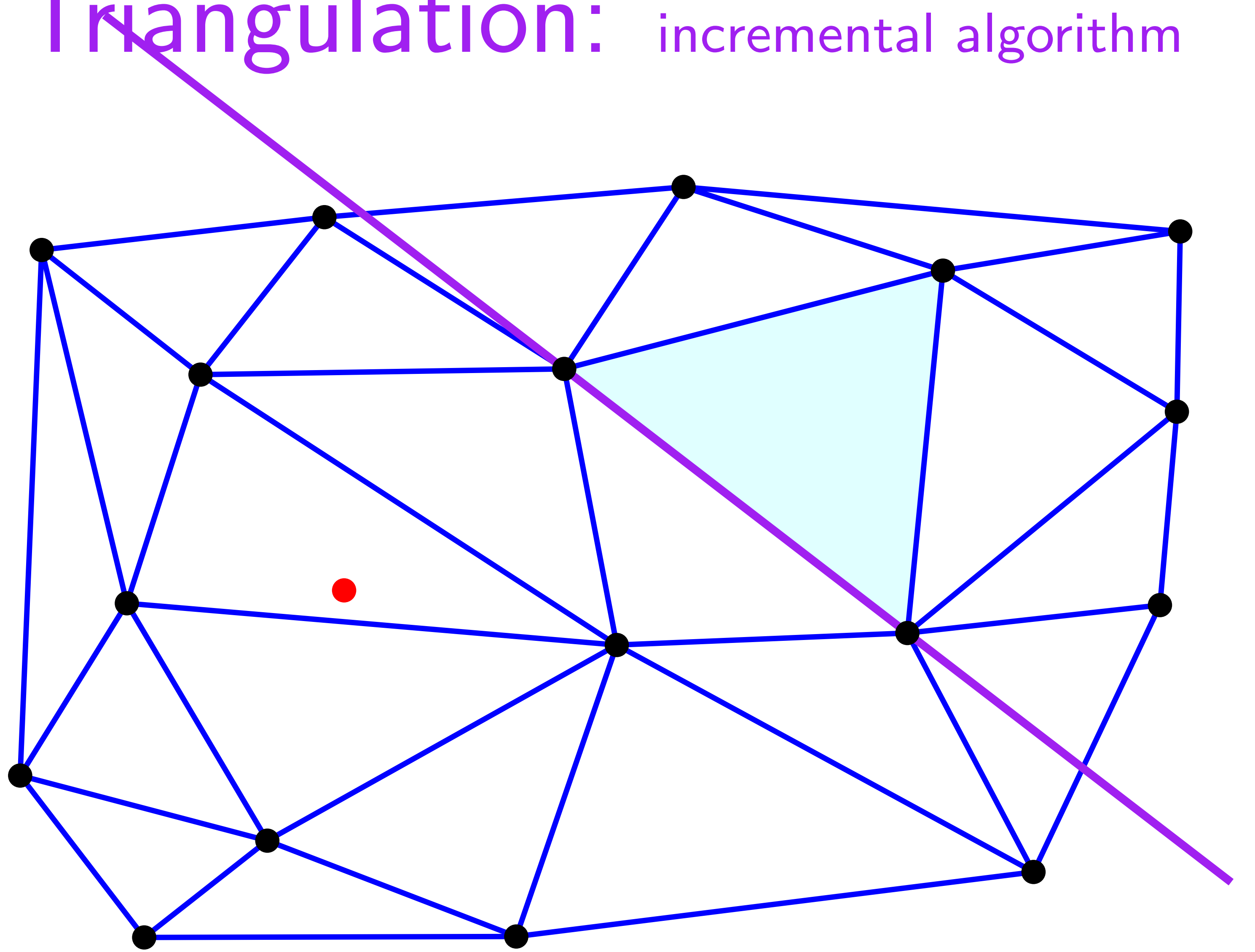


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

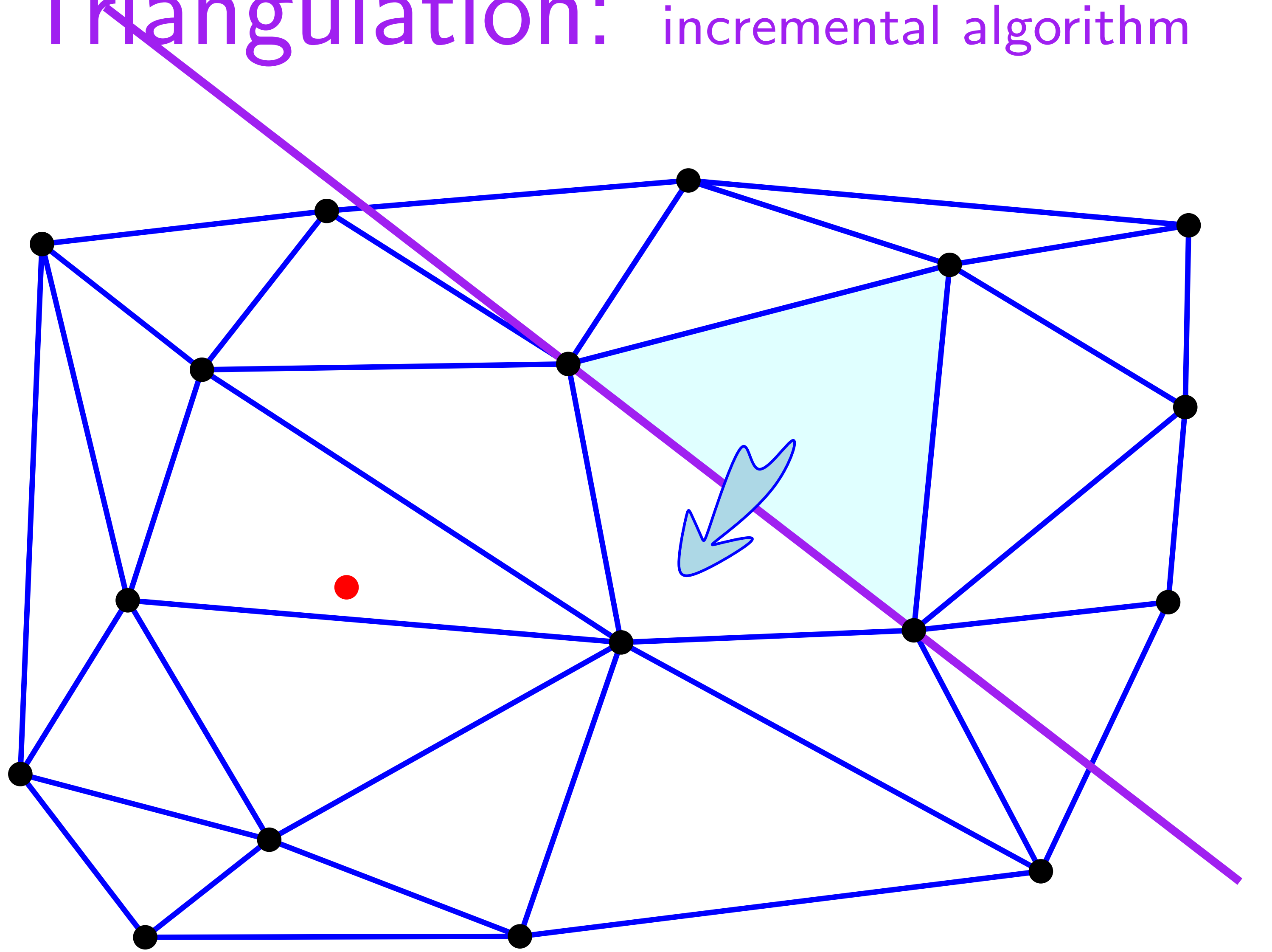


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

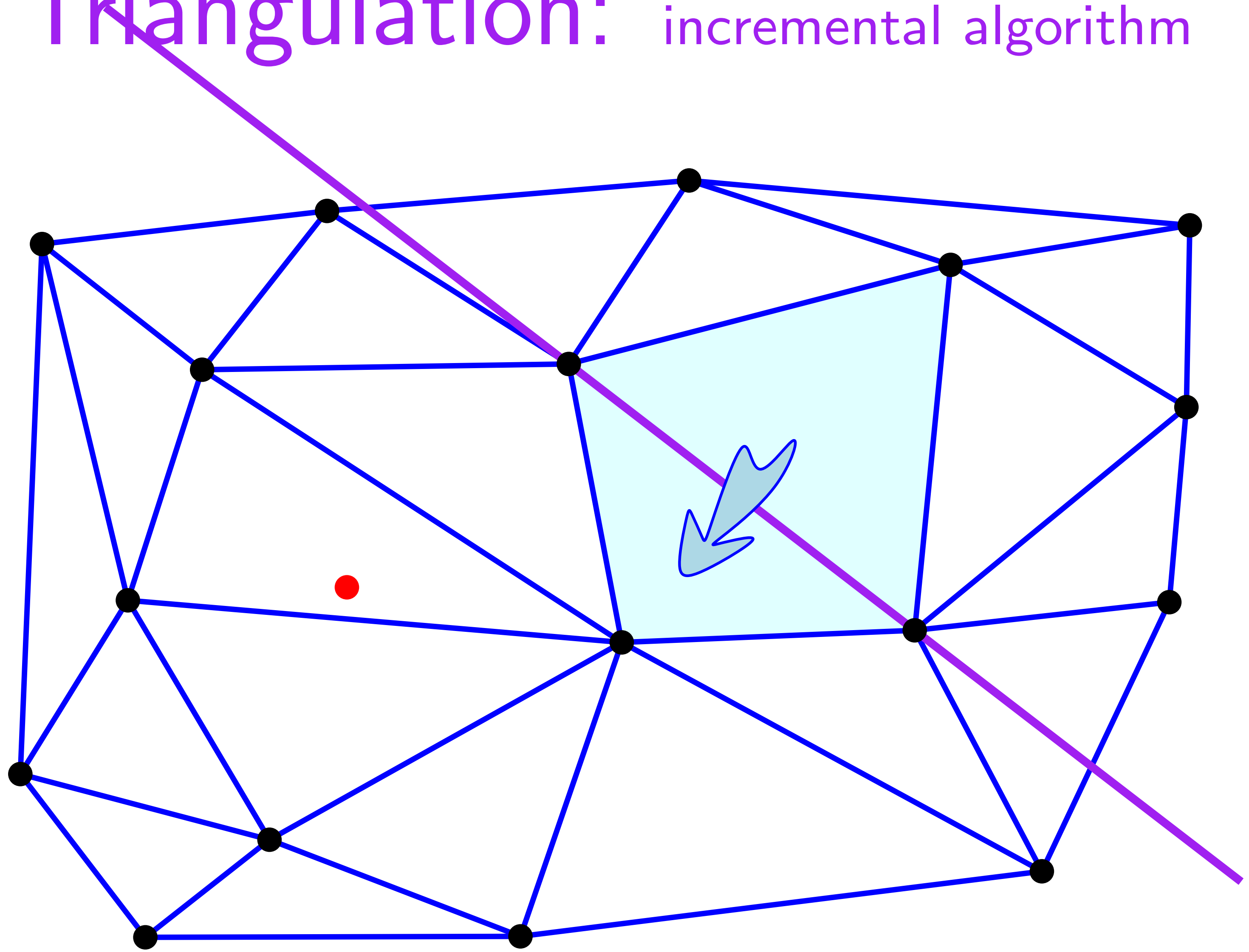


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

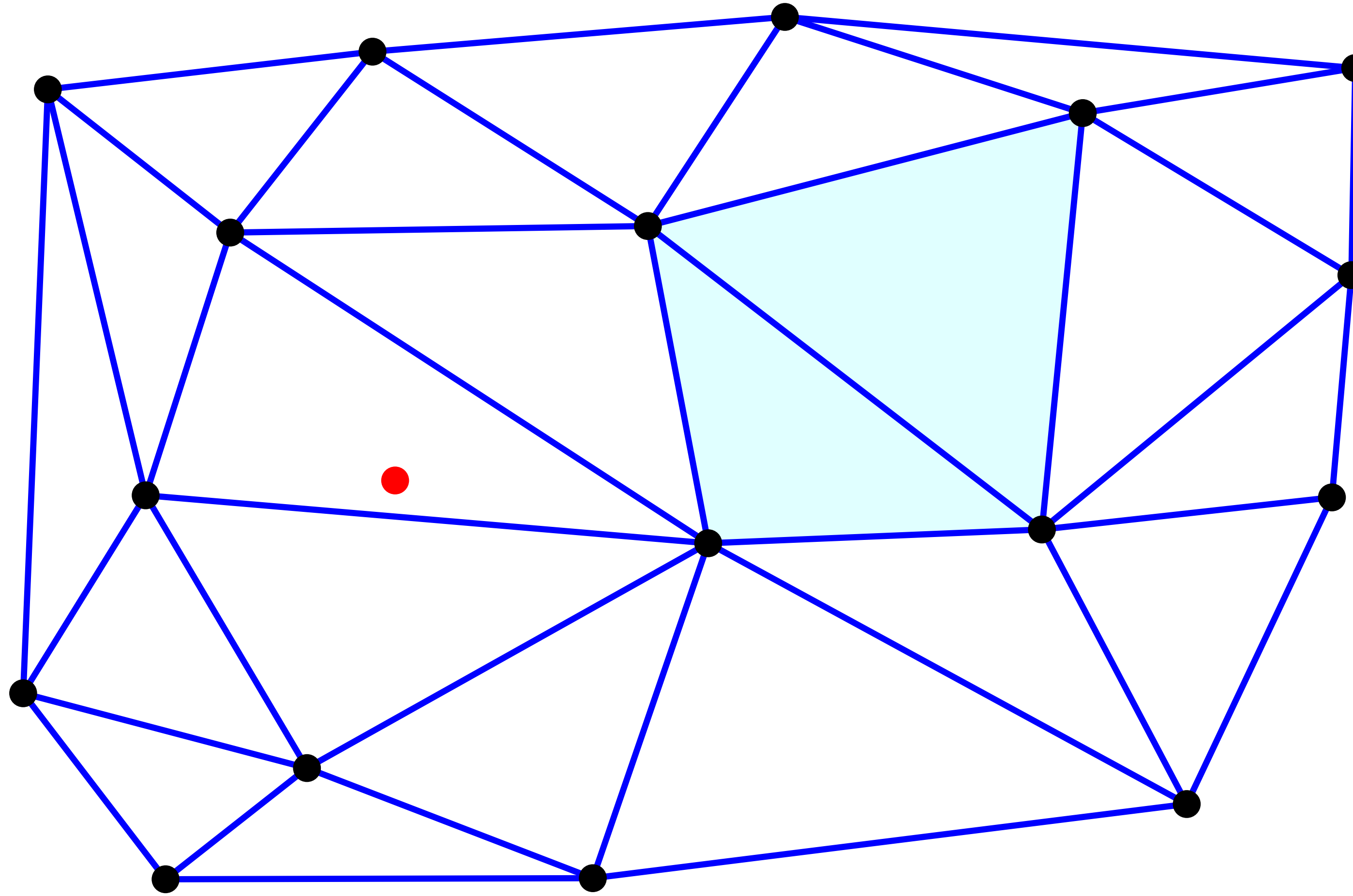


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

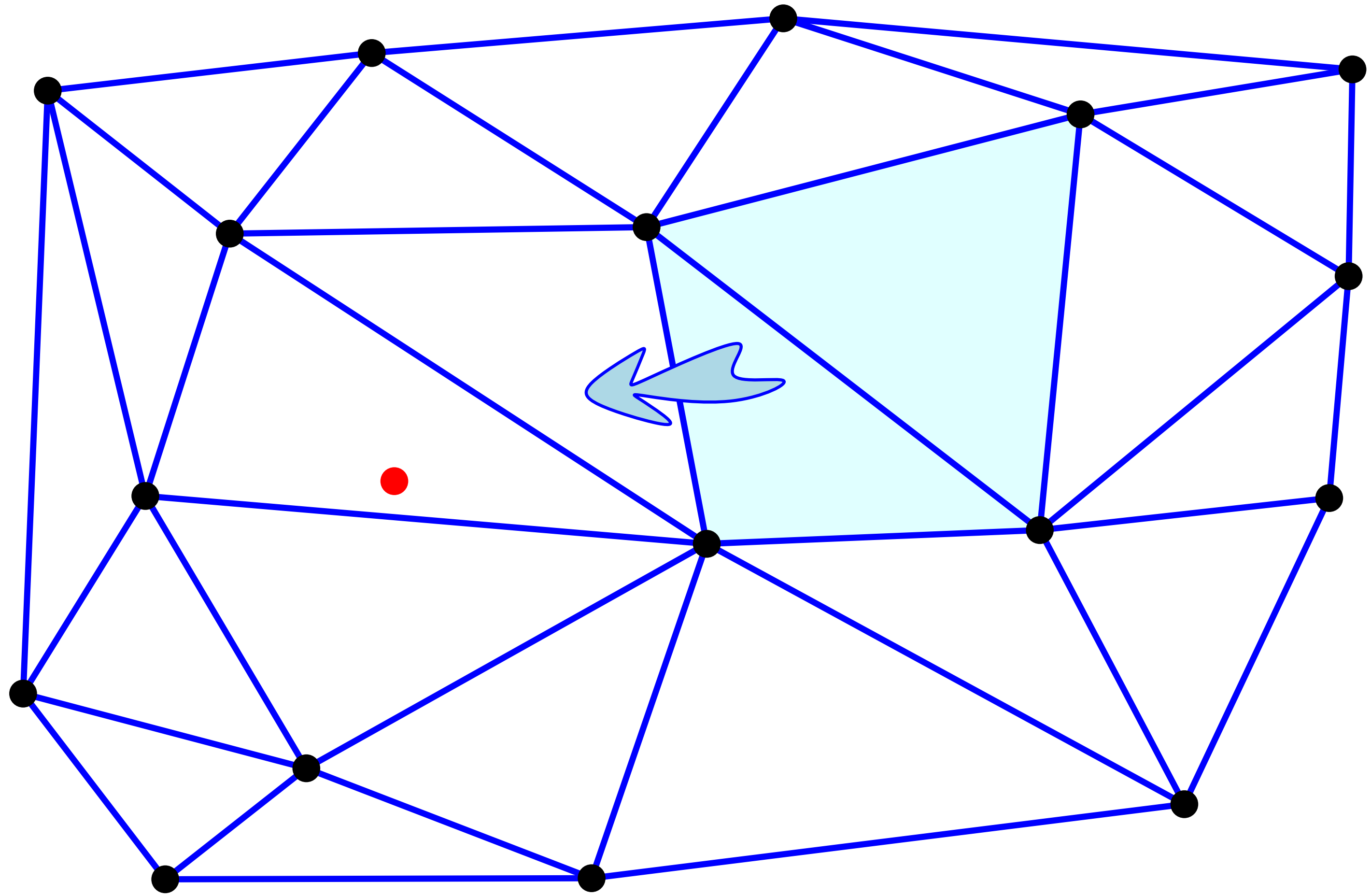


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

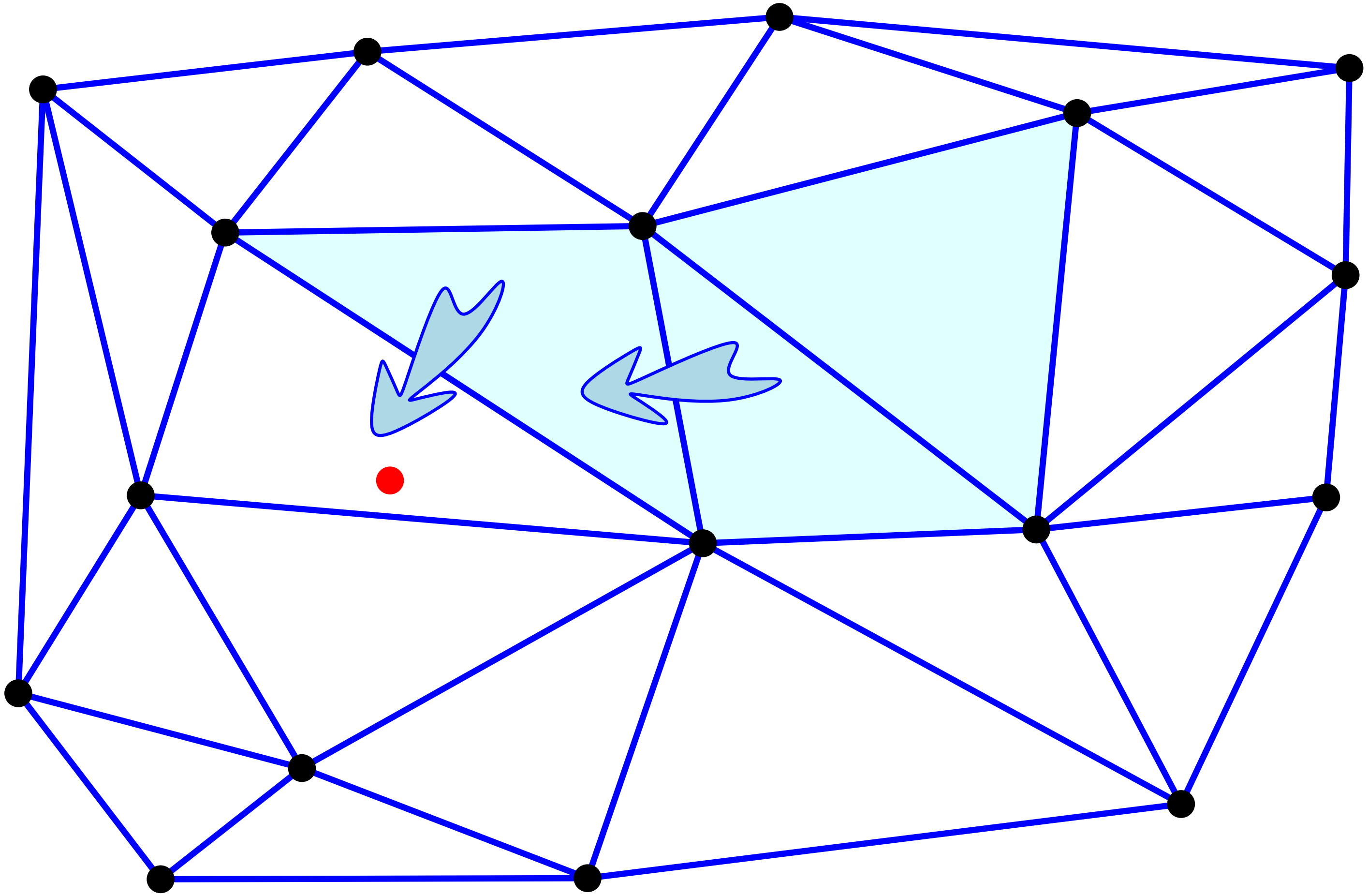


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

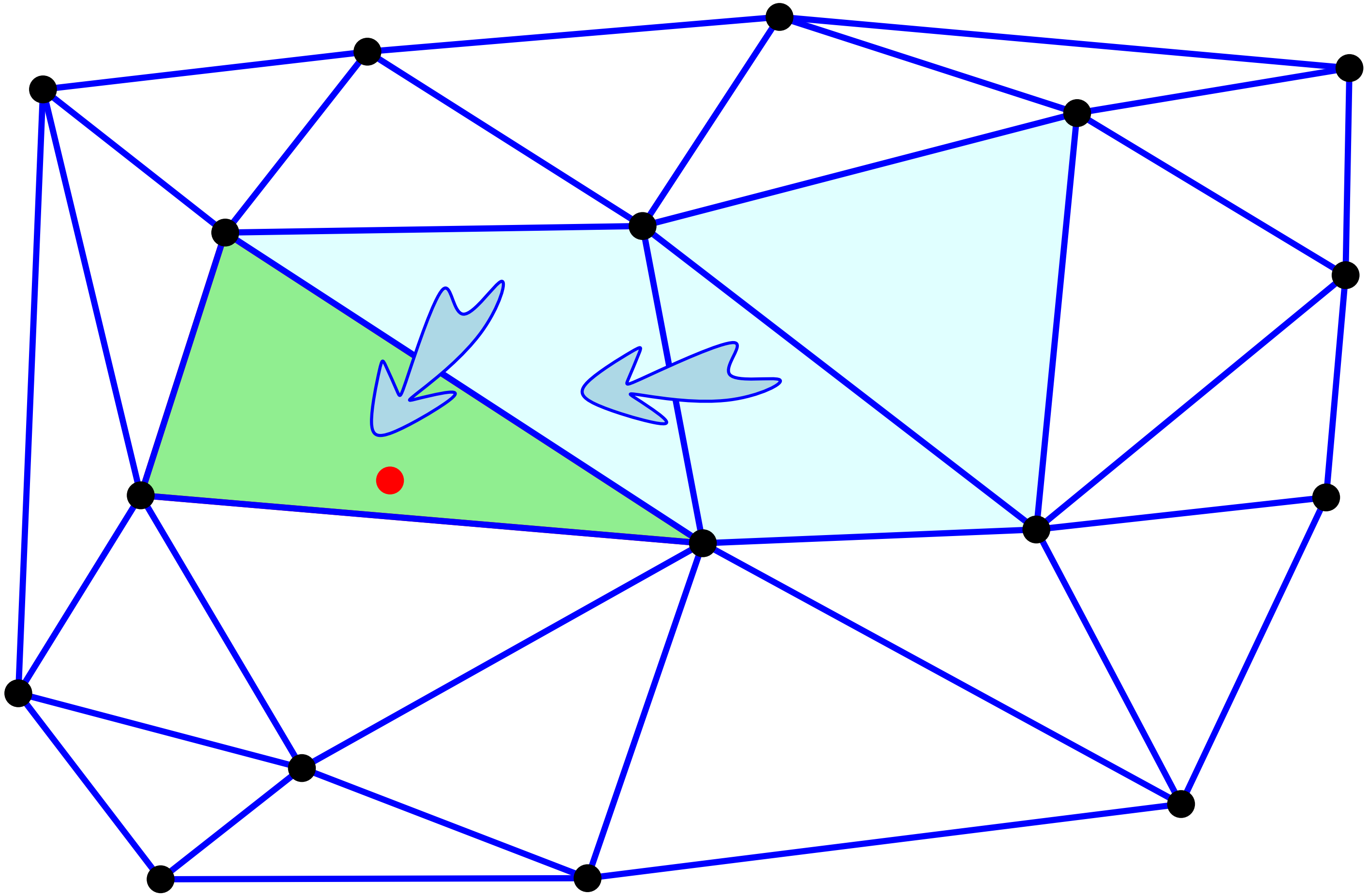


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

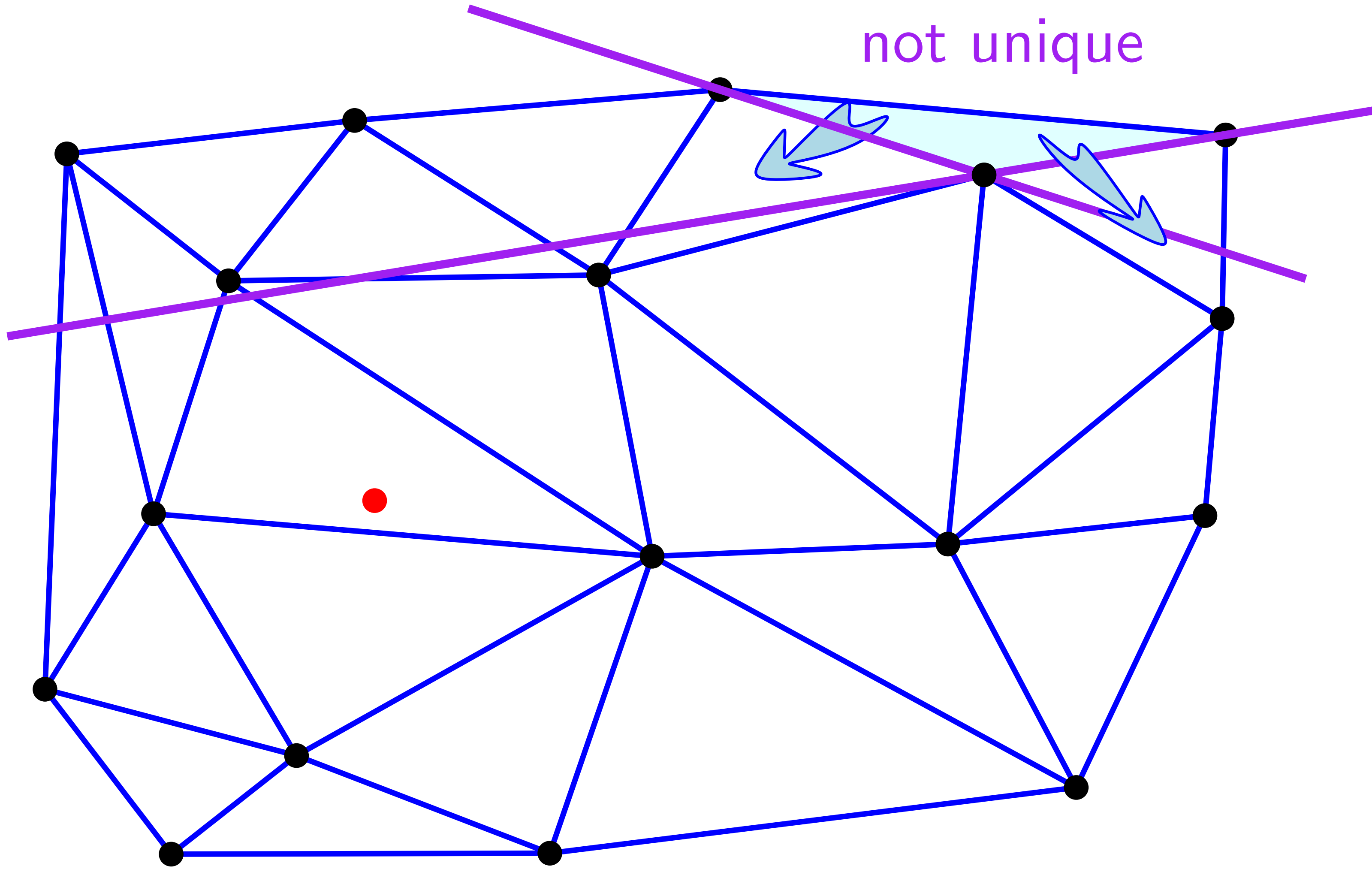


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate

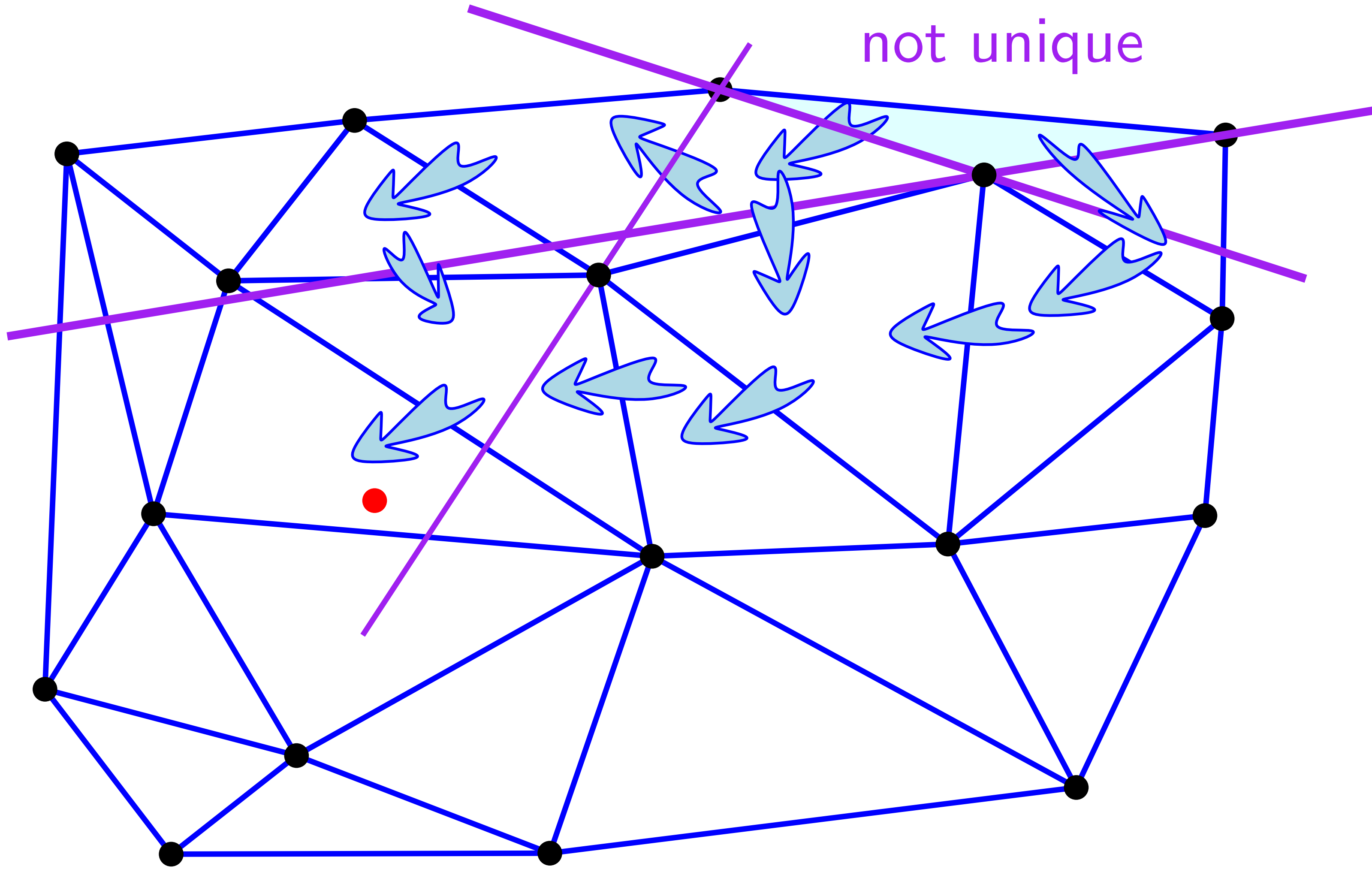


e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

New point

Locate



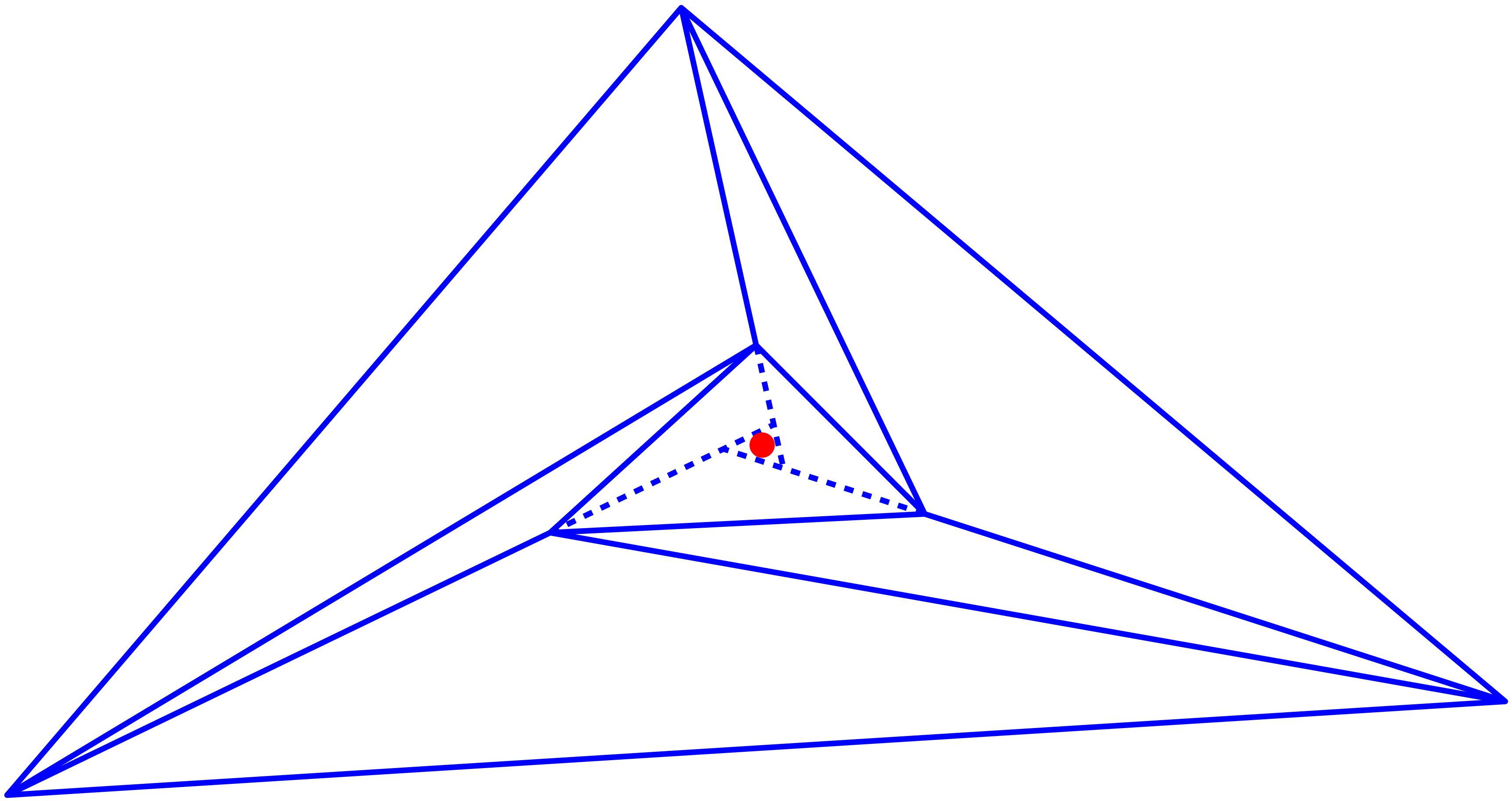
e.g.: visibility walk

Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?

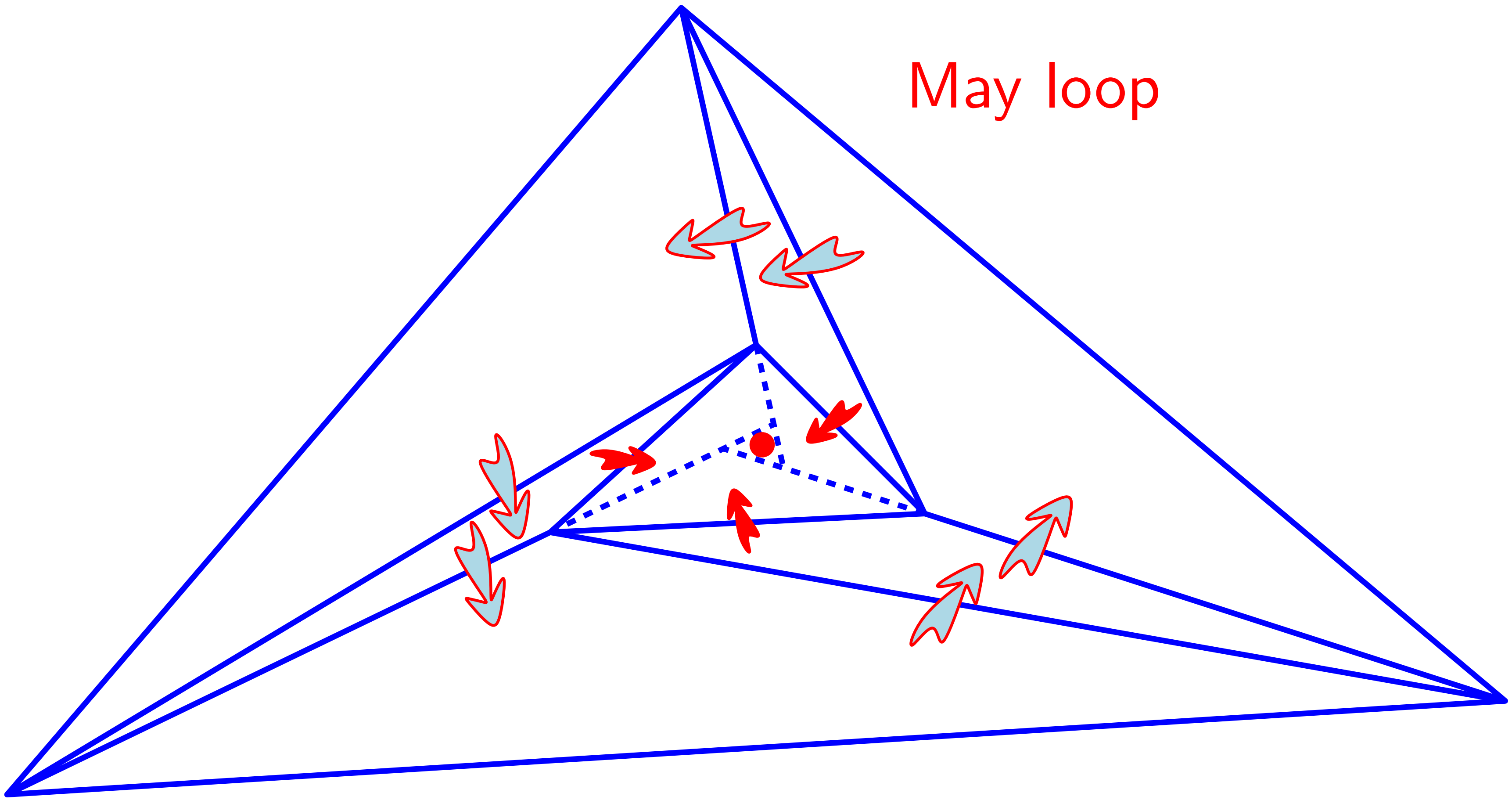
Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?



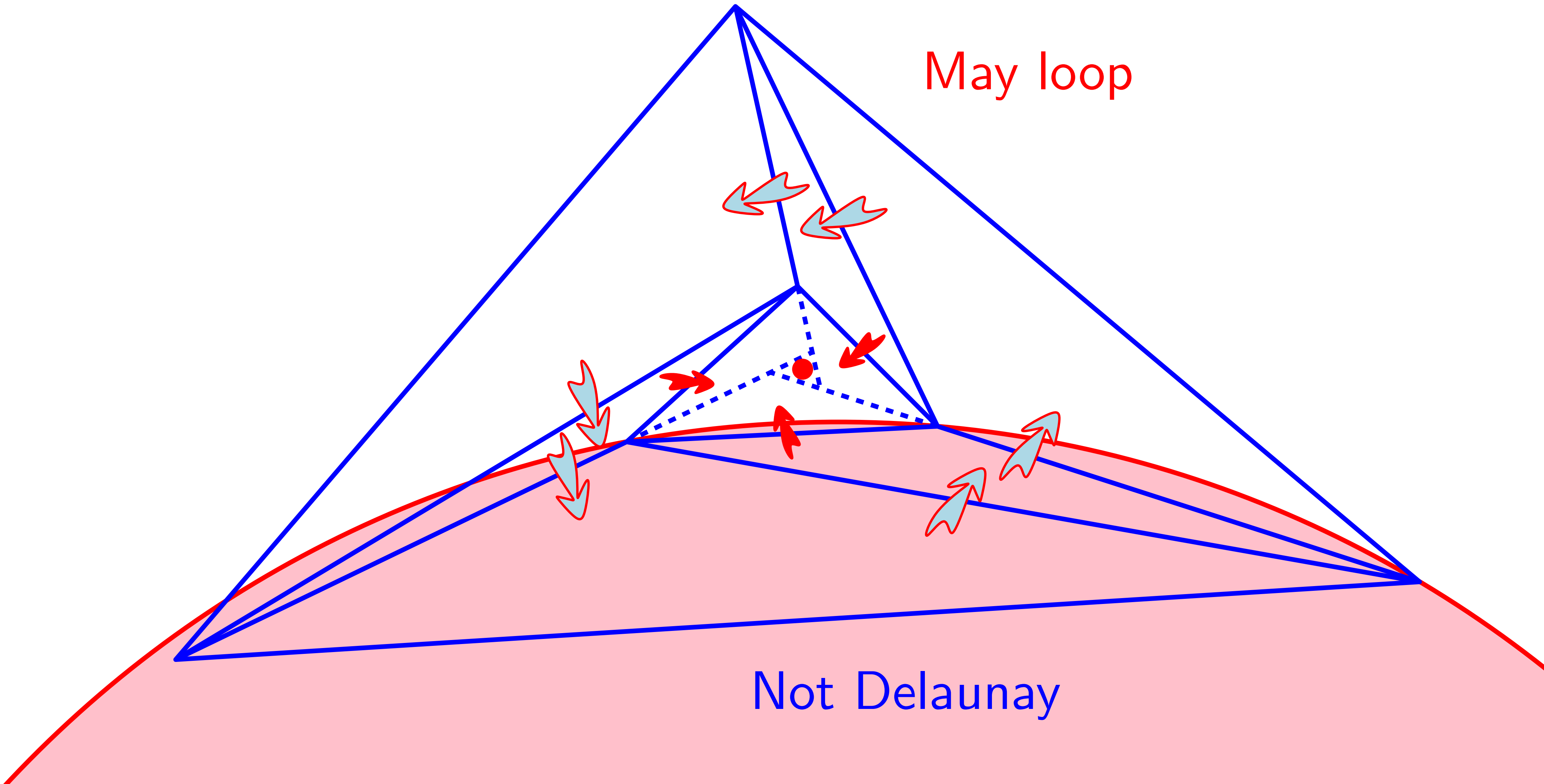
Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?



Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?



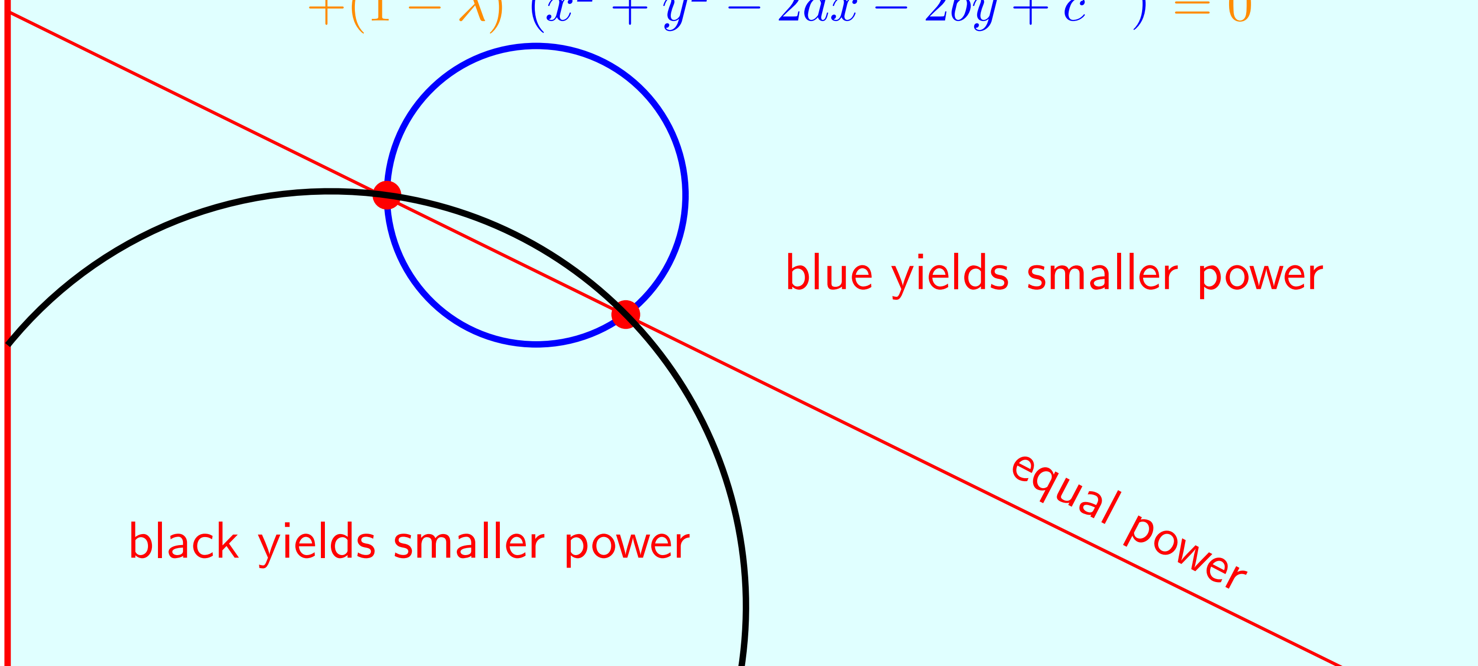
Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?

Delaunay Triangulation: pencils of circles

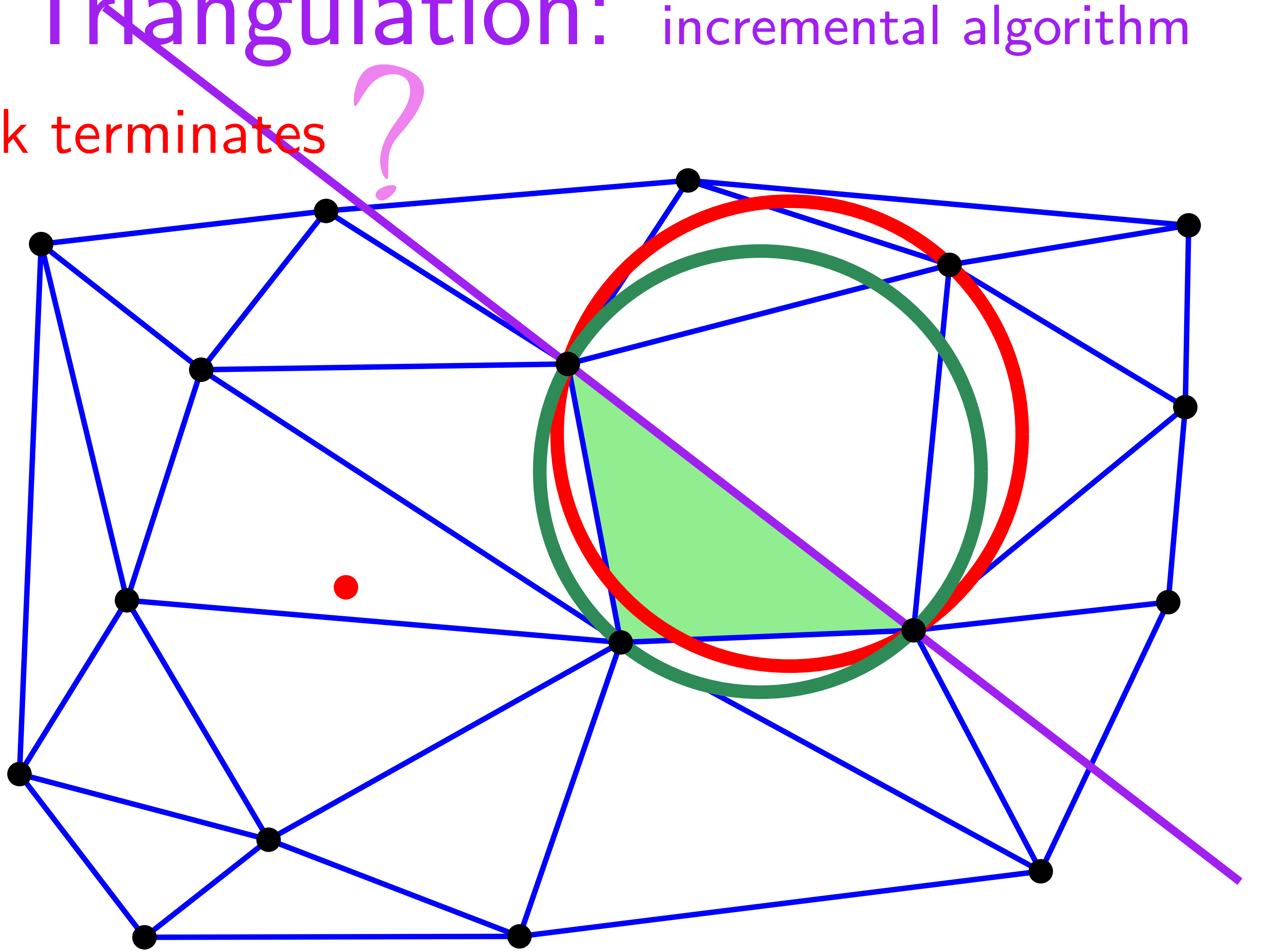
Power of a point w.r.t a circle

$$\lambda \quad (x^2 + y^2 - 2a'x - 2b'y + c')$$
$$+(1-\lambda)\ (x^2 + y^2 - 2ax - 2by + c \quad) = 0$$



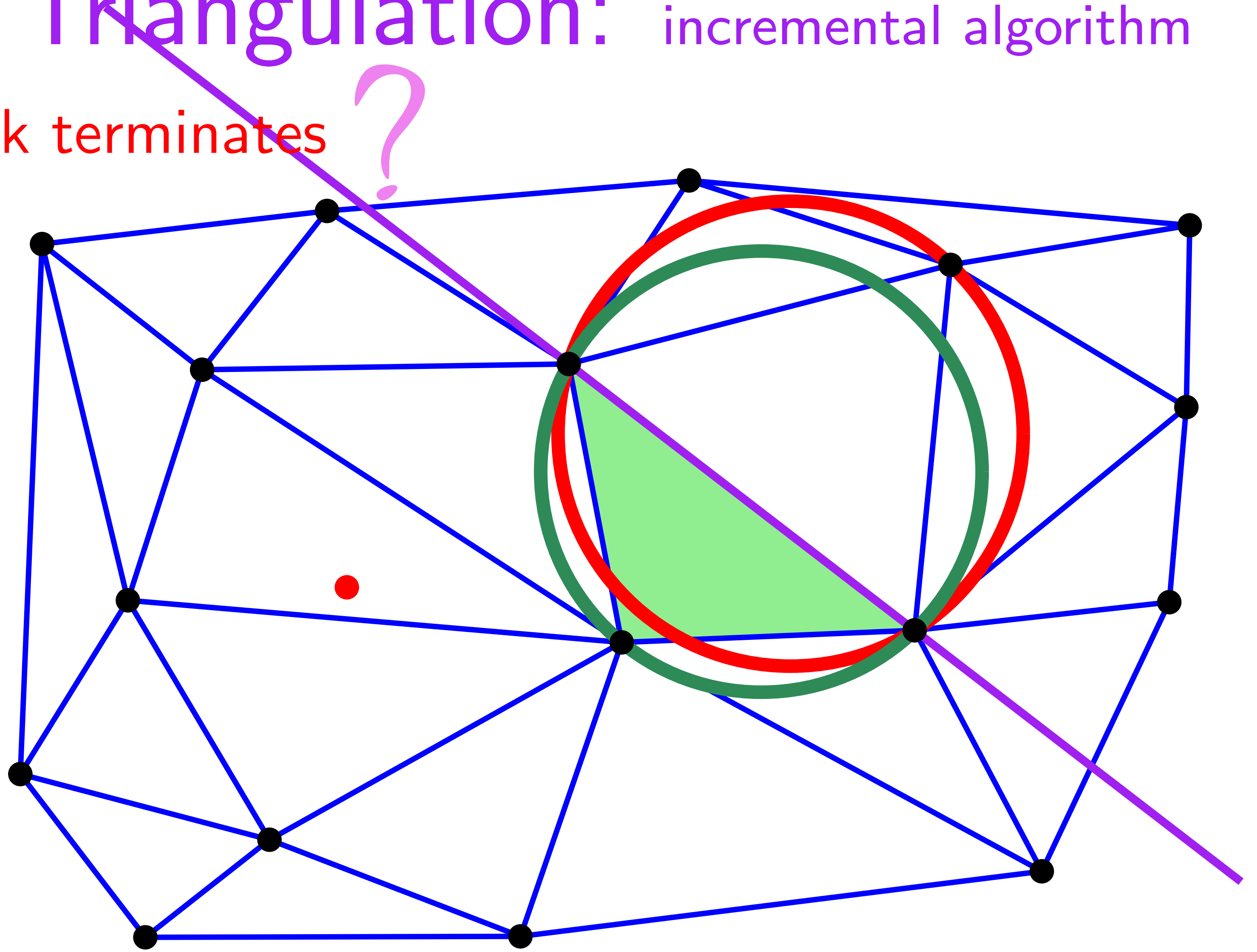
Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?



Delaunay Triangulation: incremental algorithm

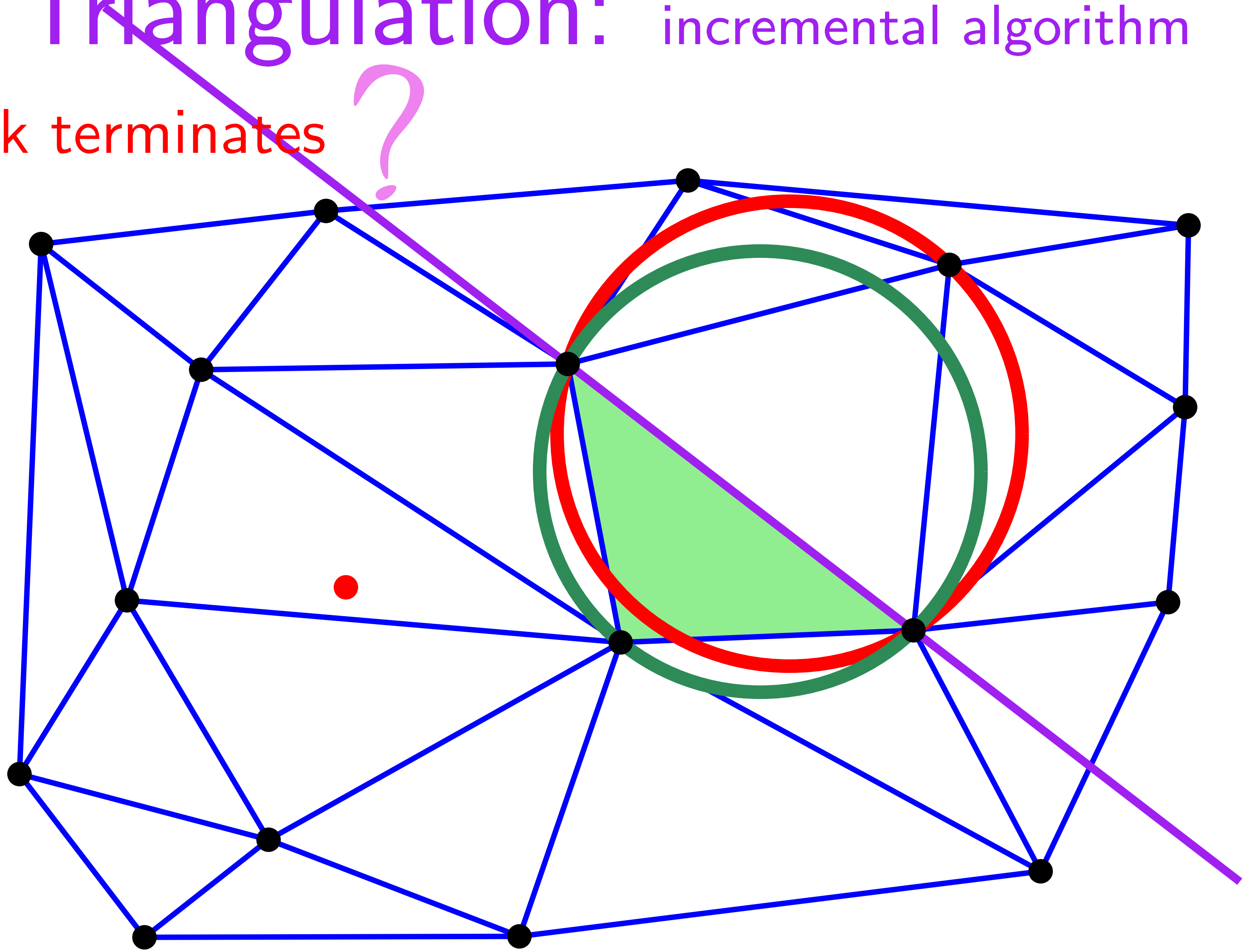
Visibility walk terminates ?



Green power $<$ Red power

Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?

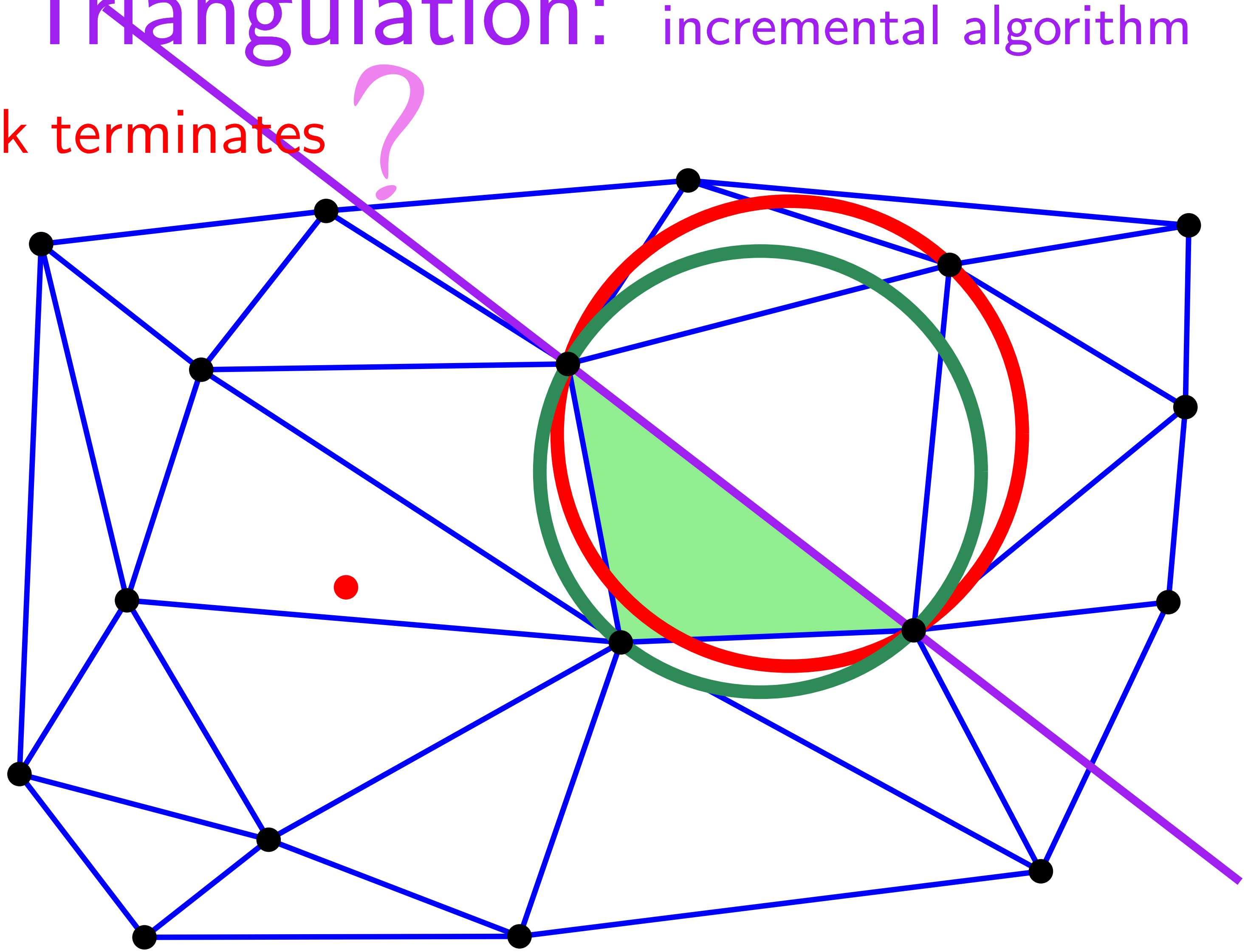


Green power $<$ Red power

Power decreases

Delaunay Triangulation: incremental algorithm

Visibility walk terminates ?



Green power < Red power

Power decreases

Visibility walk terminates

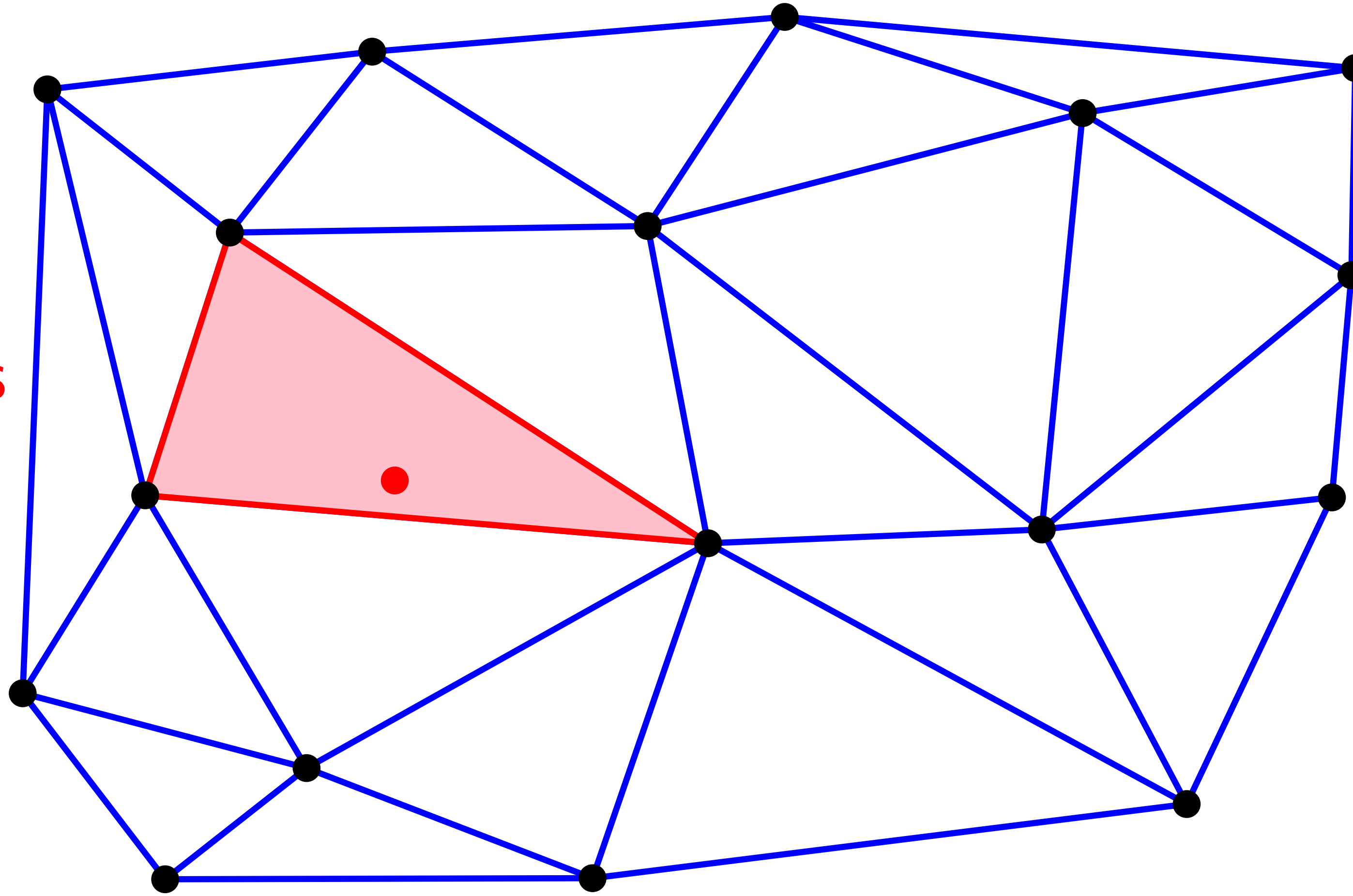
Algorithm: incremental

Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

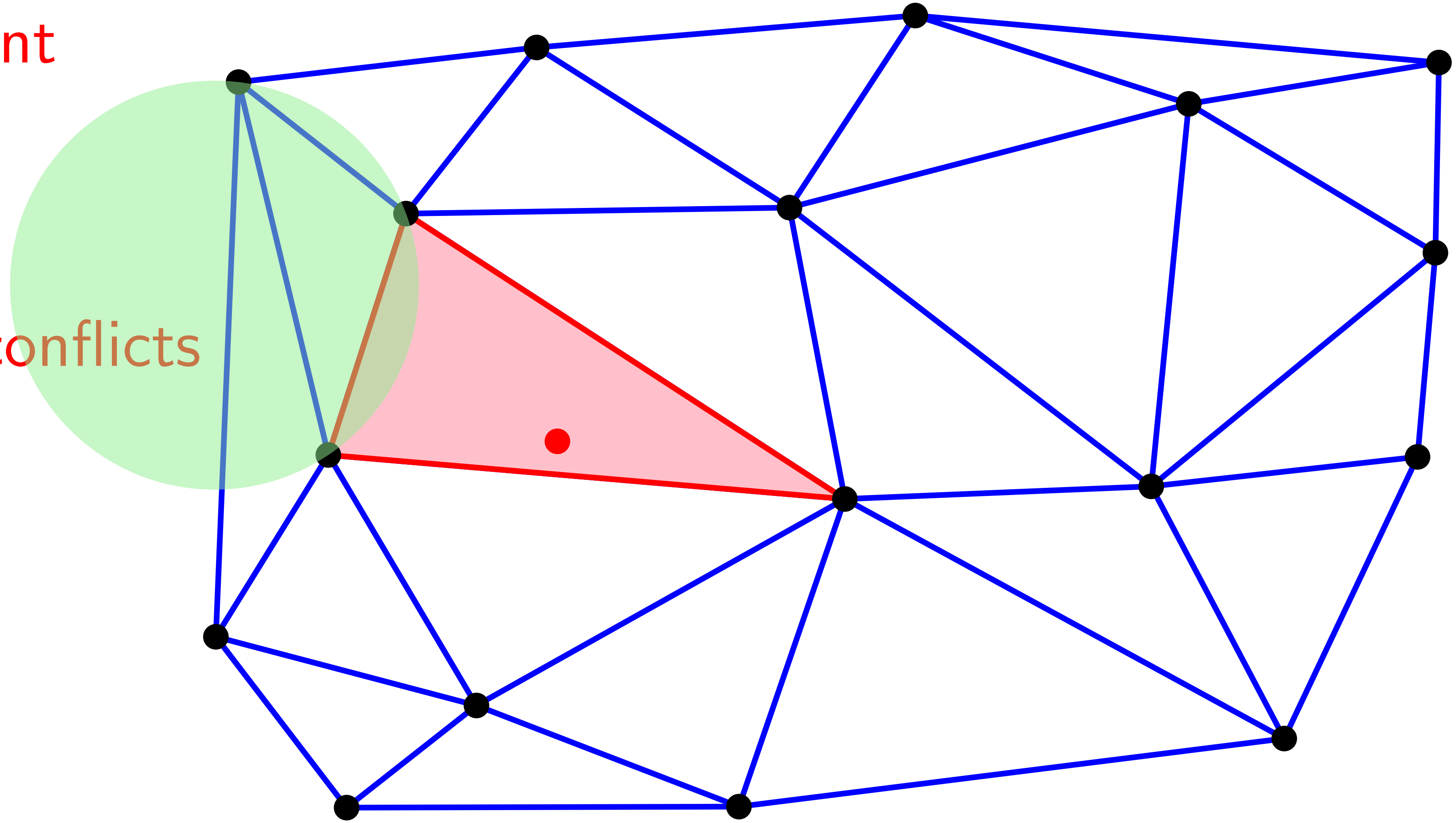


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

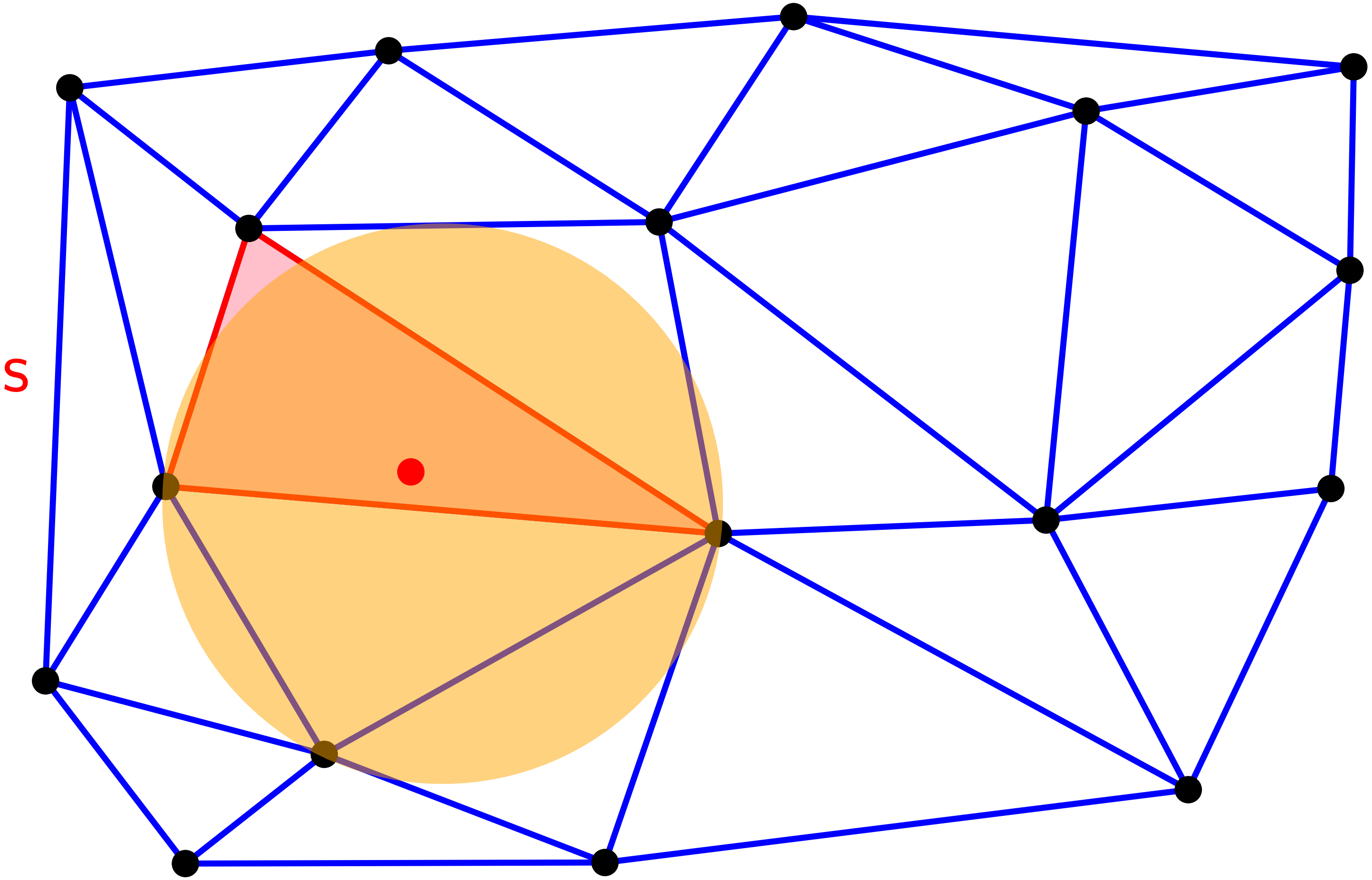


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

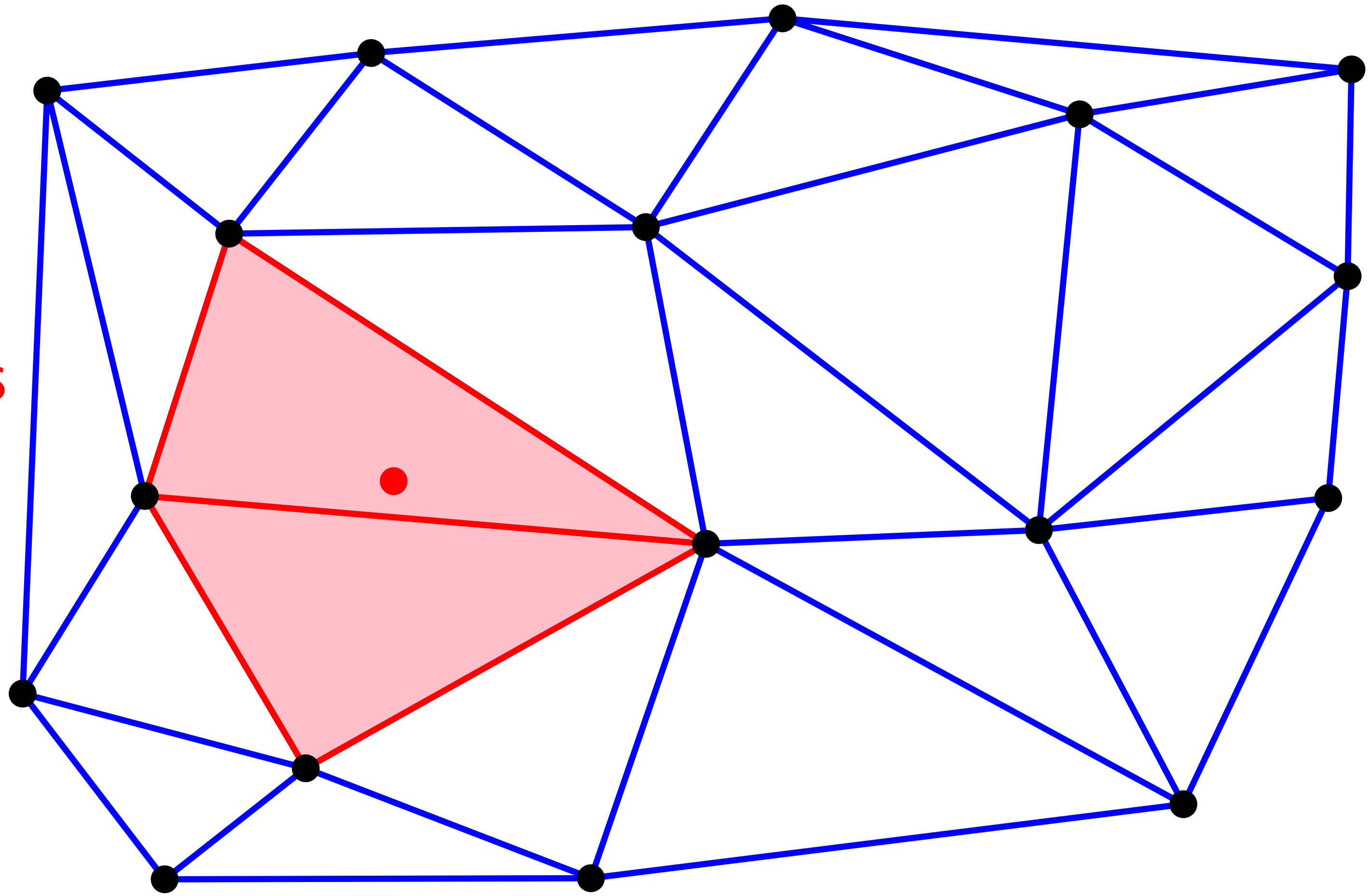


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

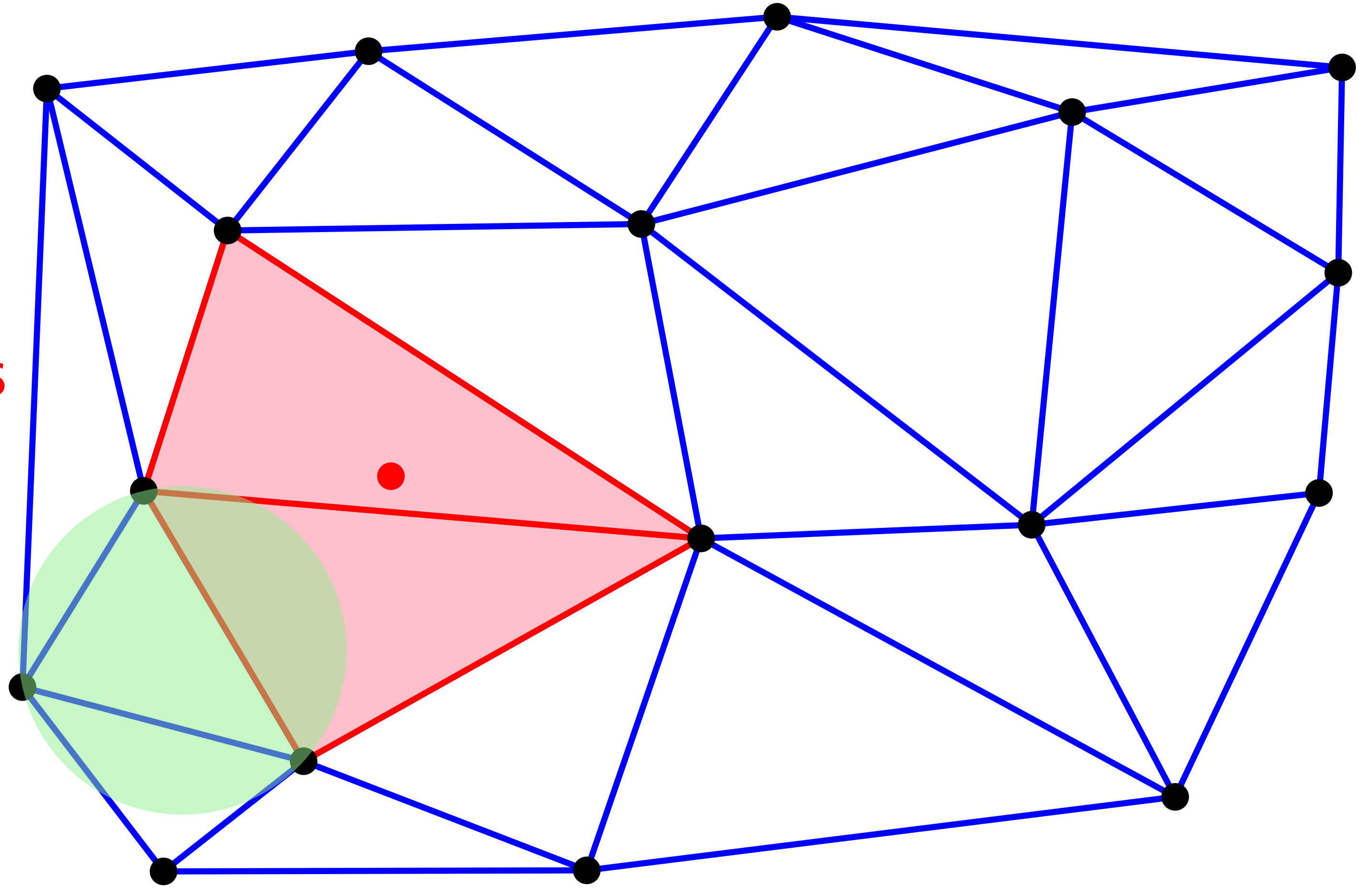


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

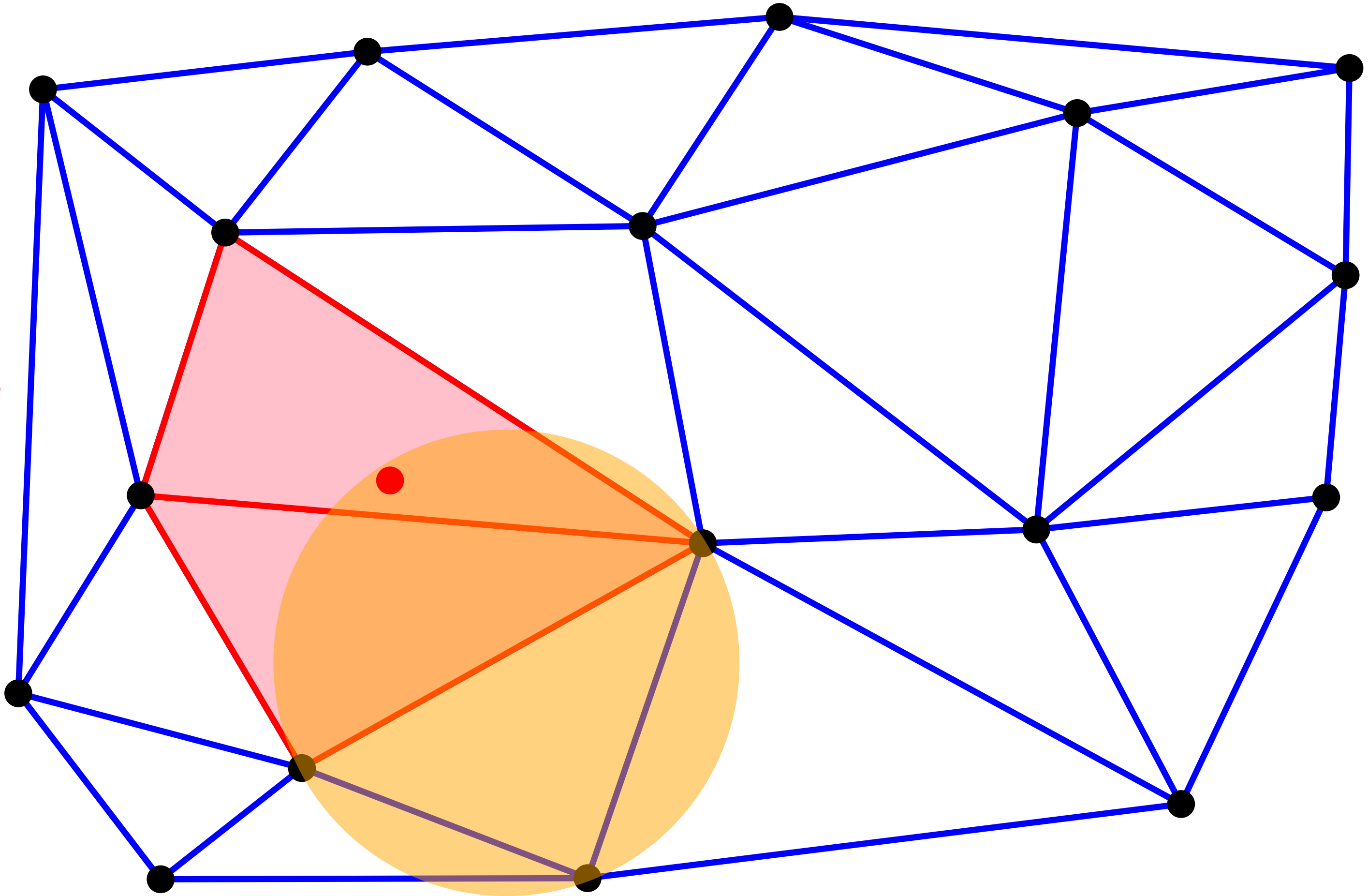


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

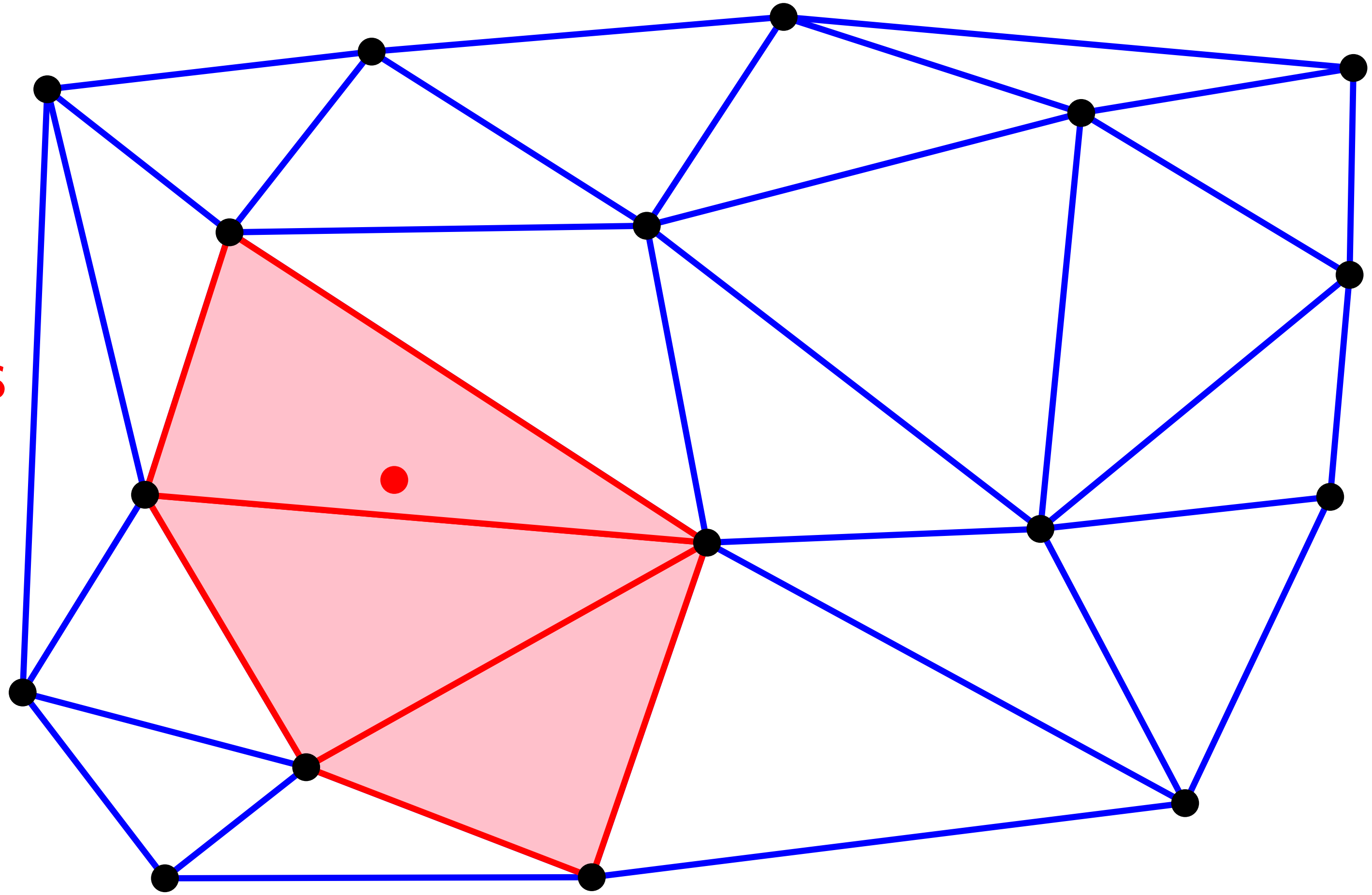


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

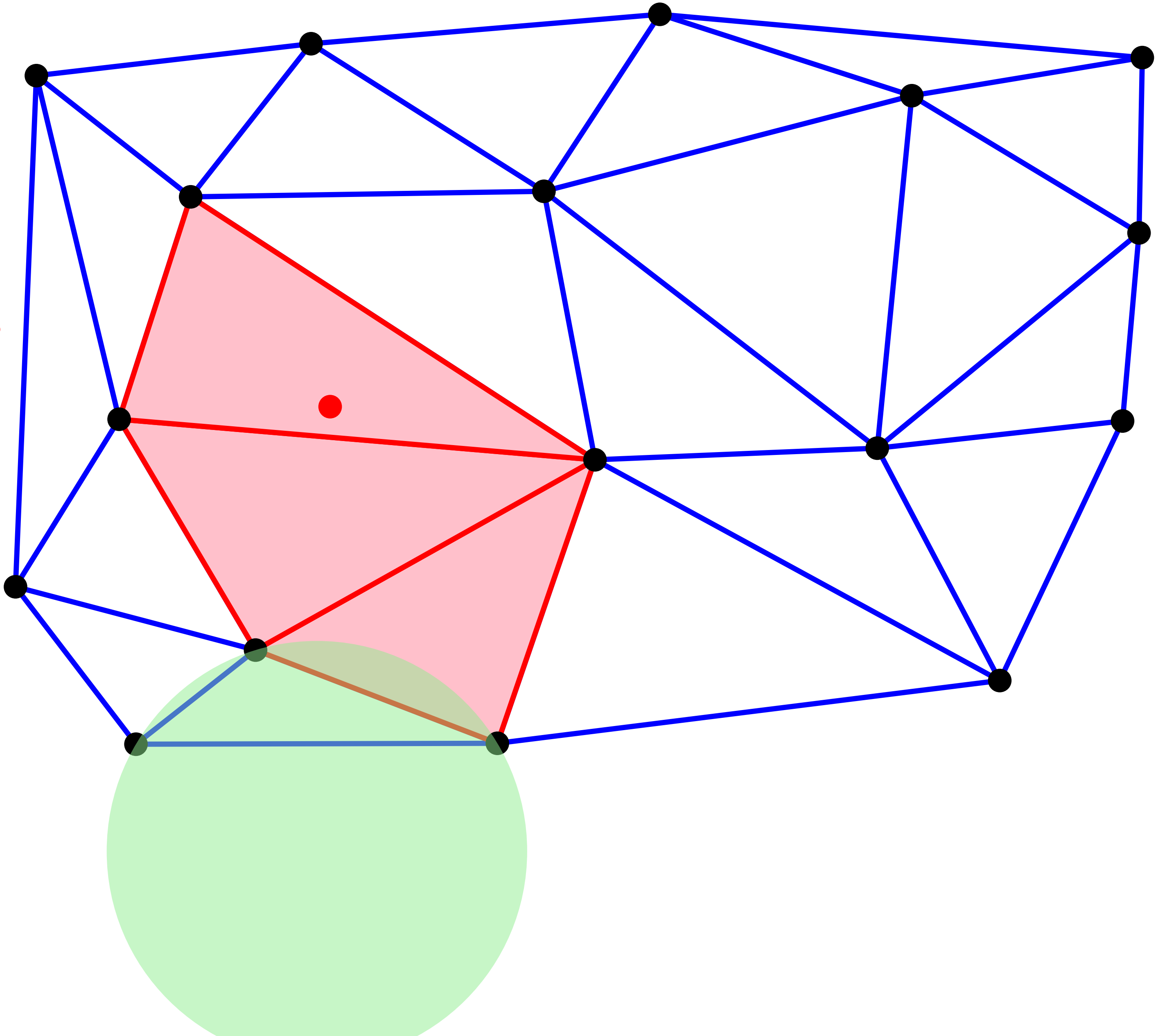


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

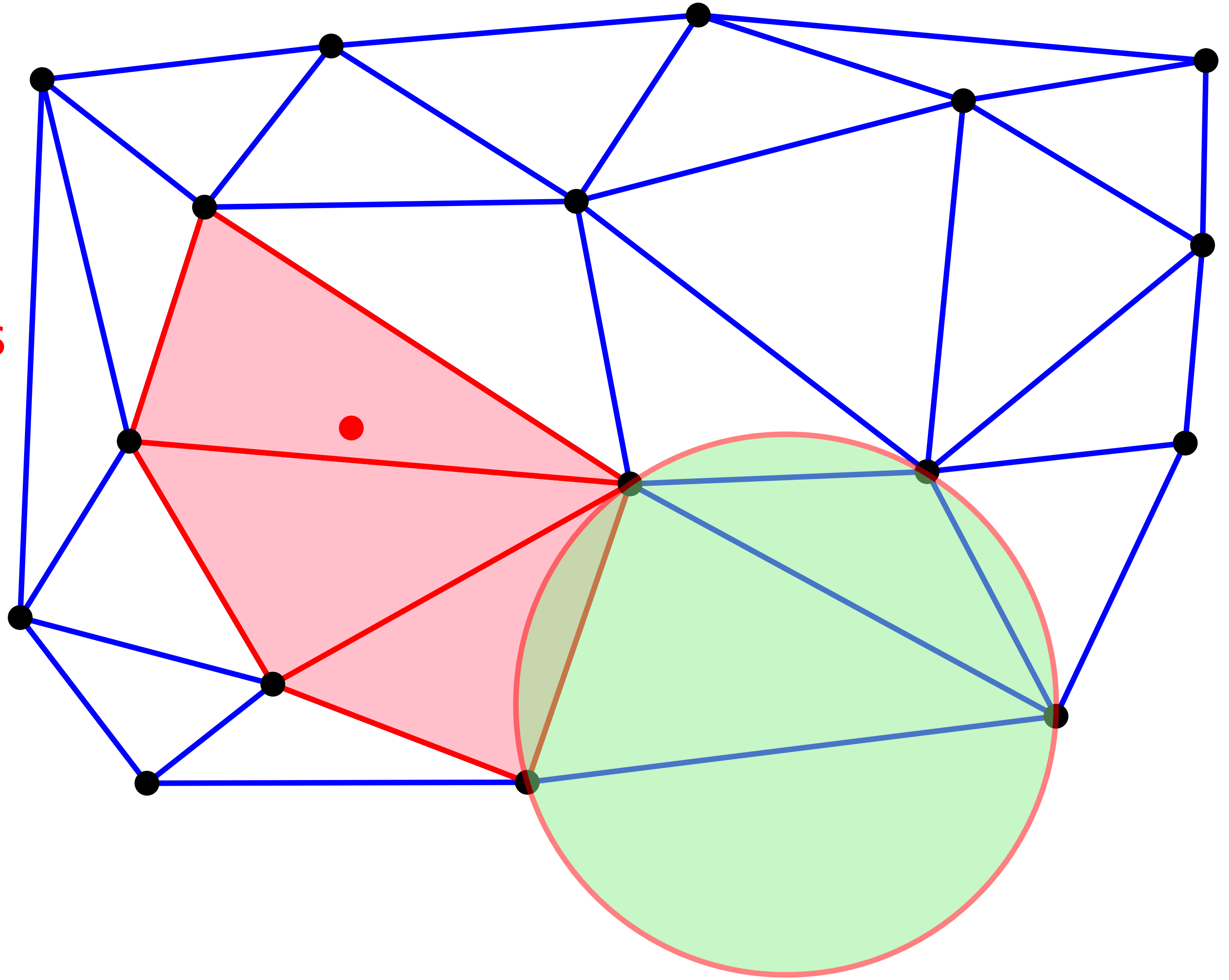


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

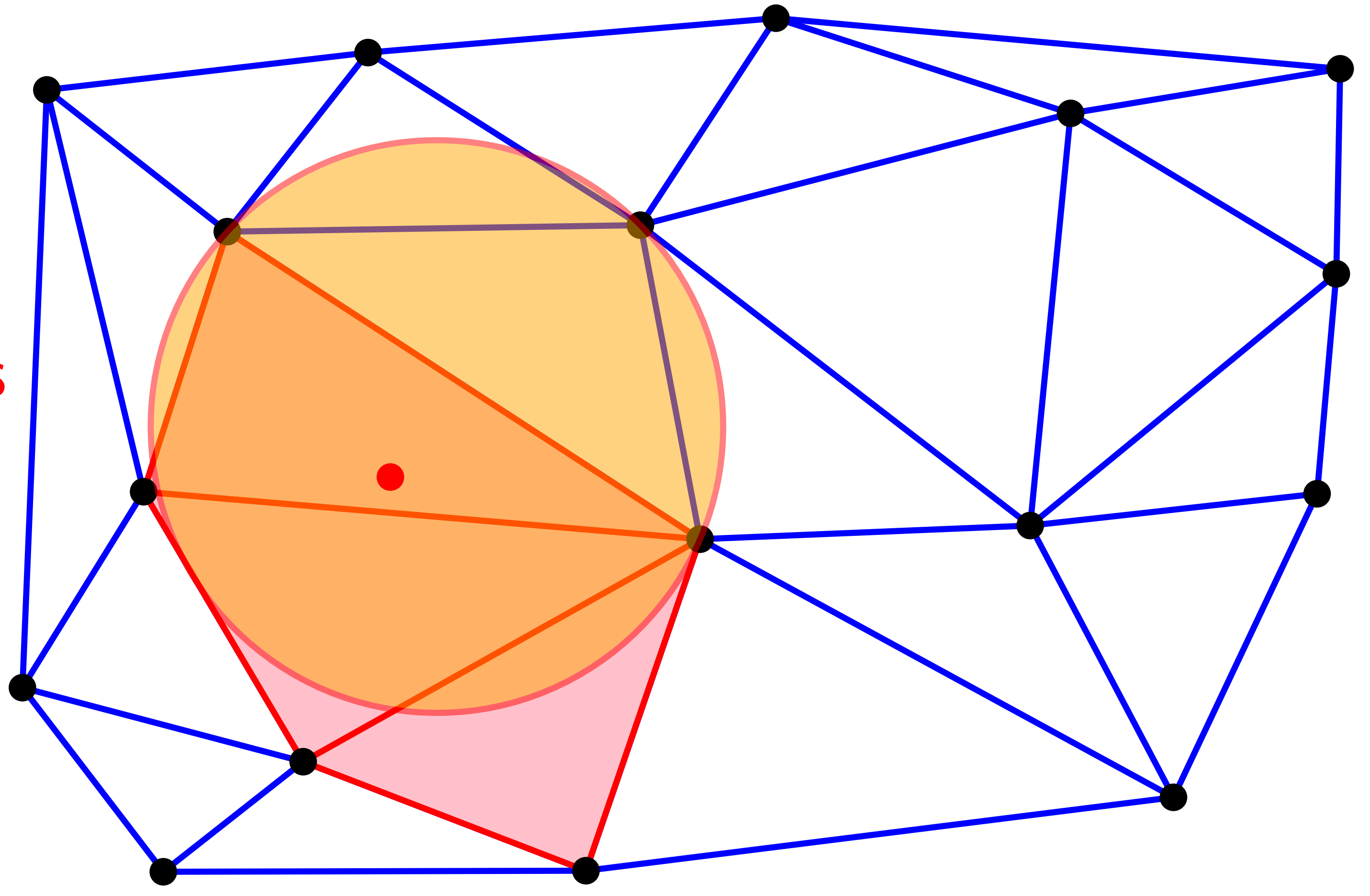


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

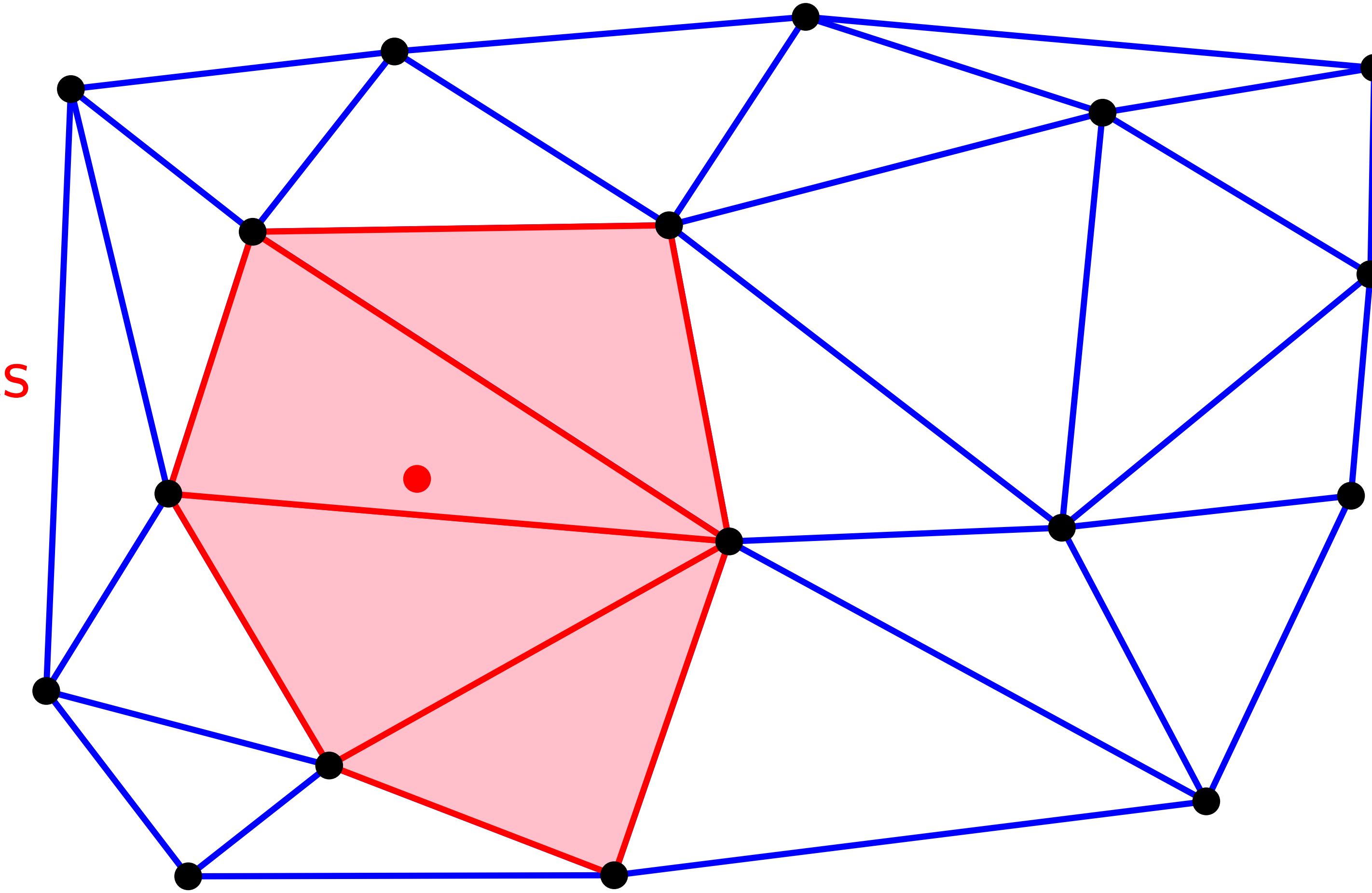


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

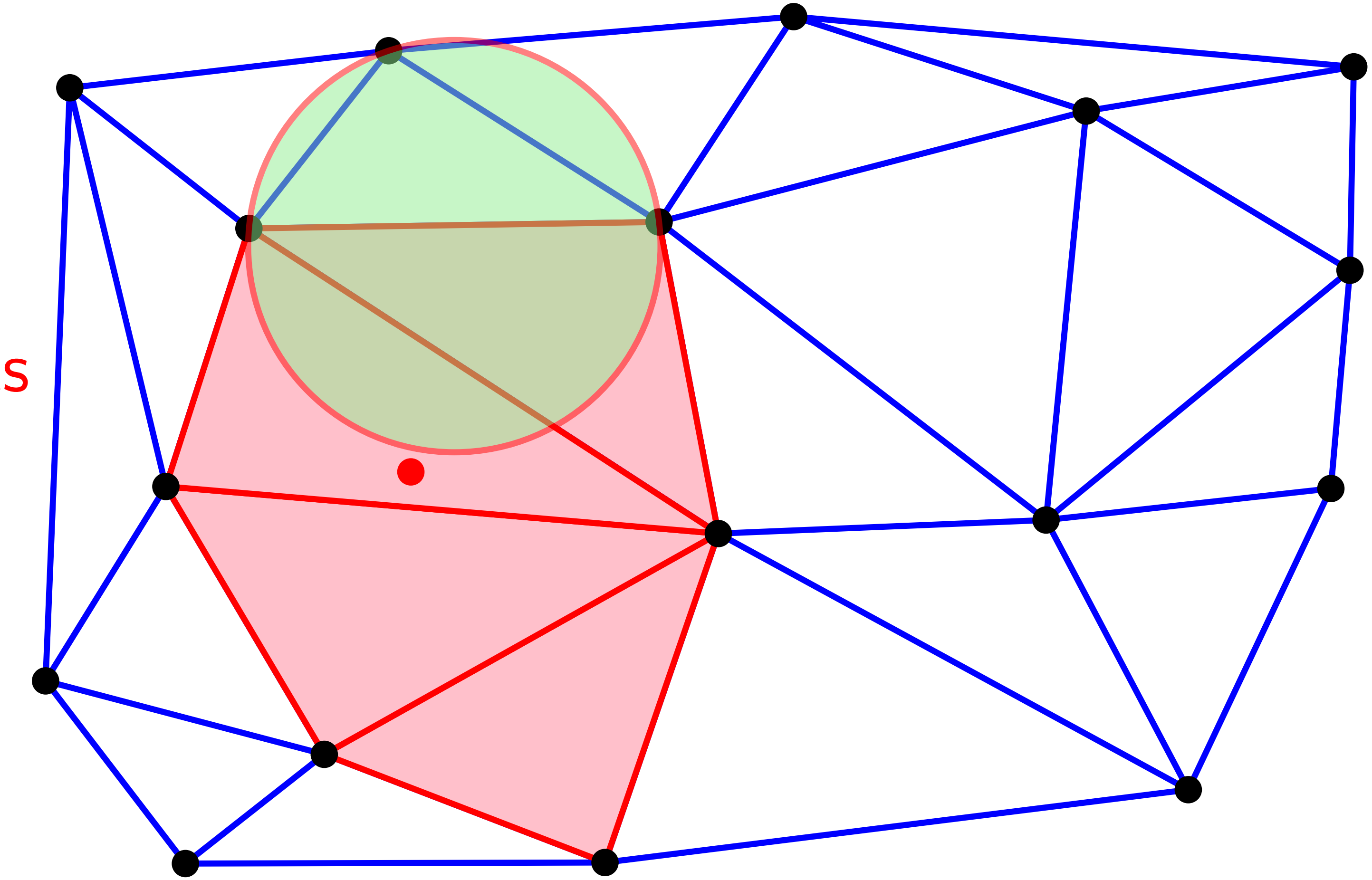


Delaunay Triangulation: incremental algorithm

New point

Locate

Search conflicts

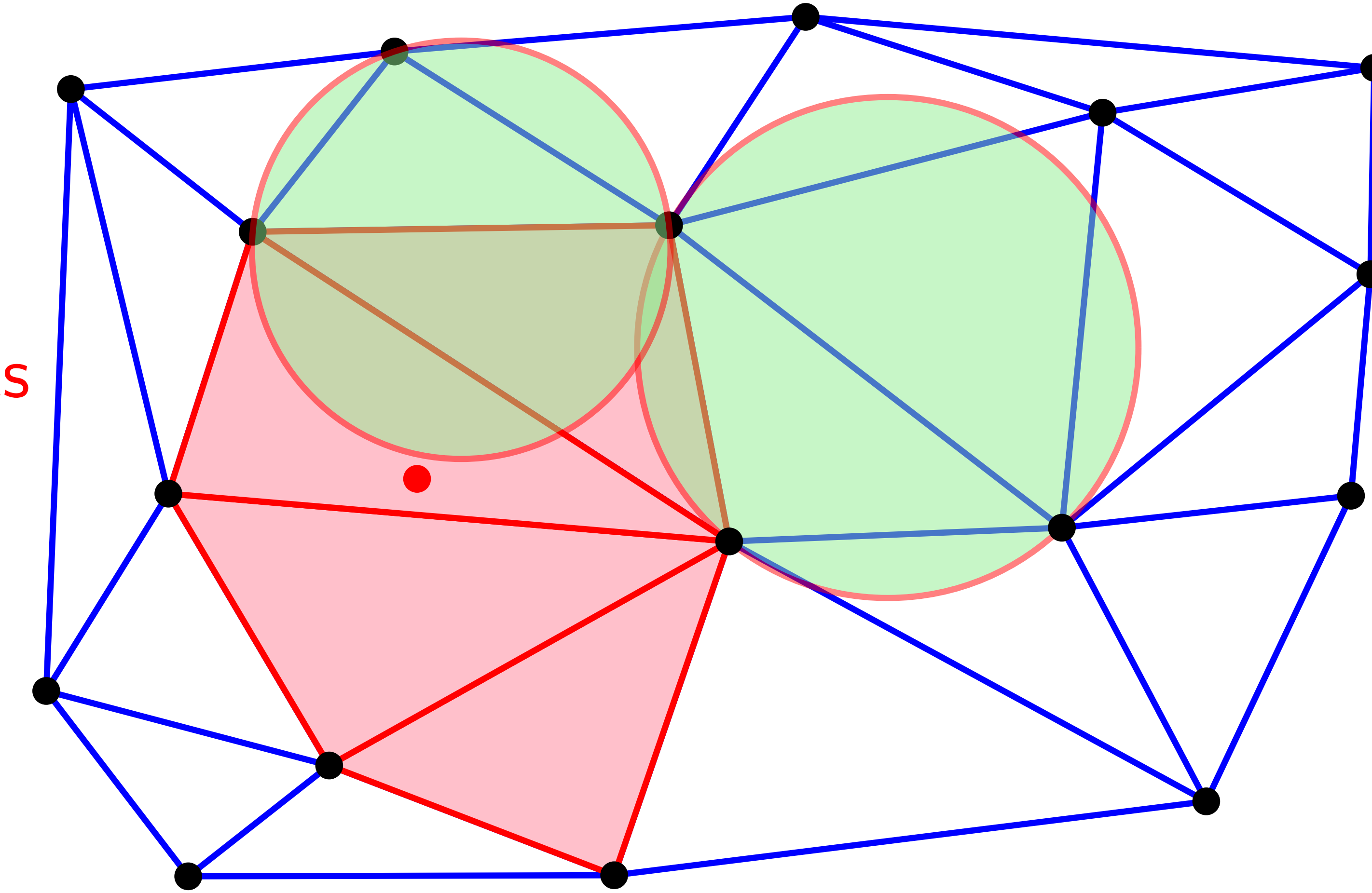


Delaunay Triangulation: incremental algorithm

New point

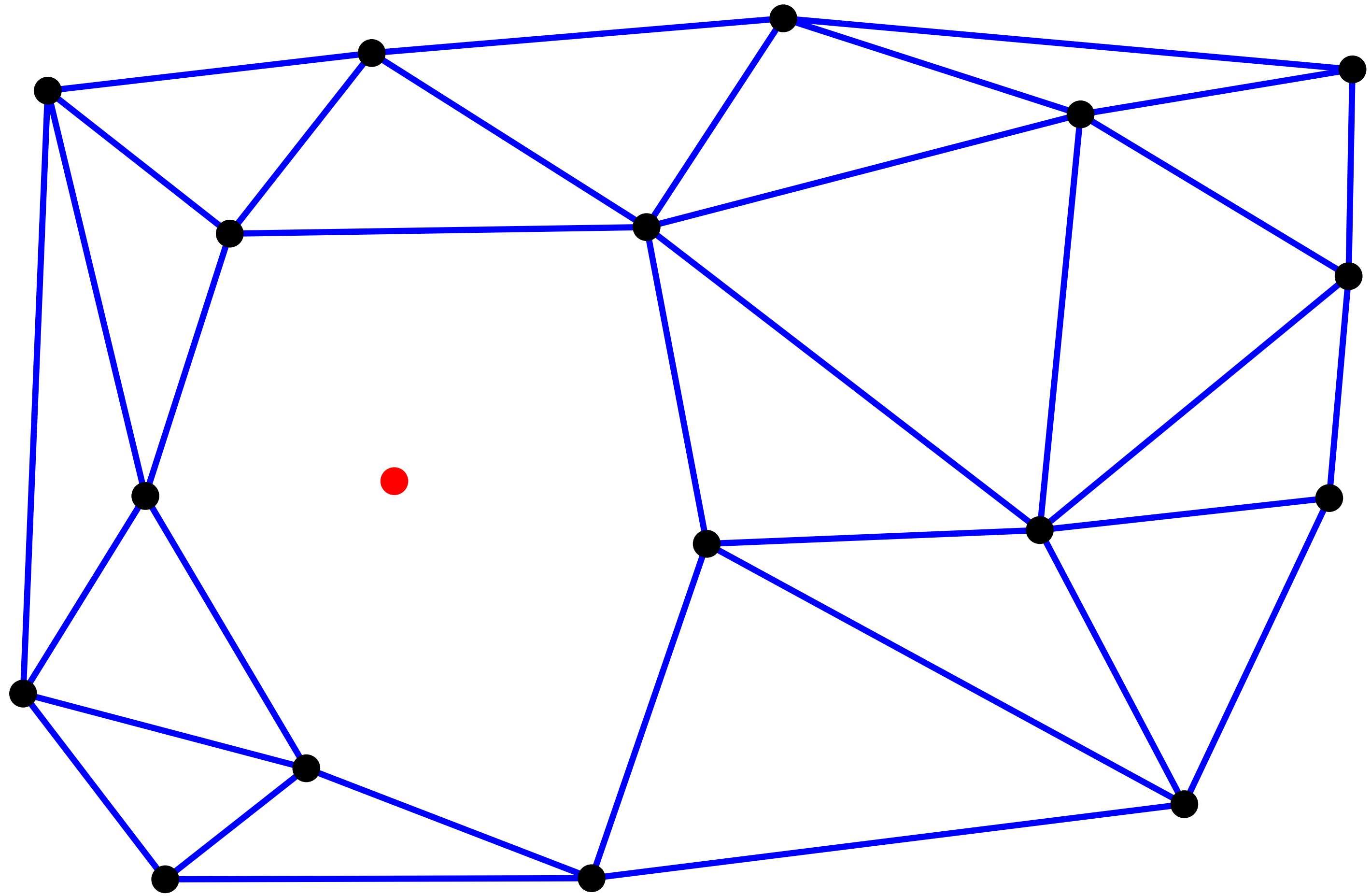
Locate

Search conflicts



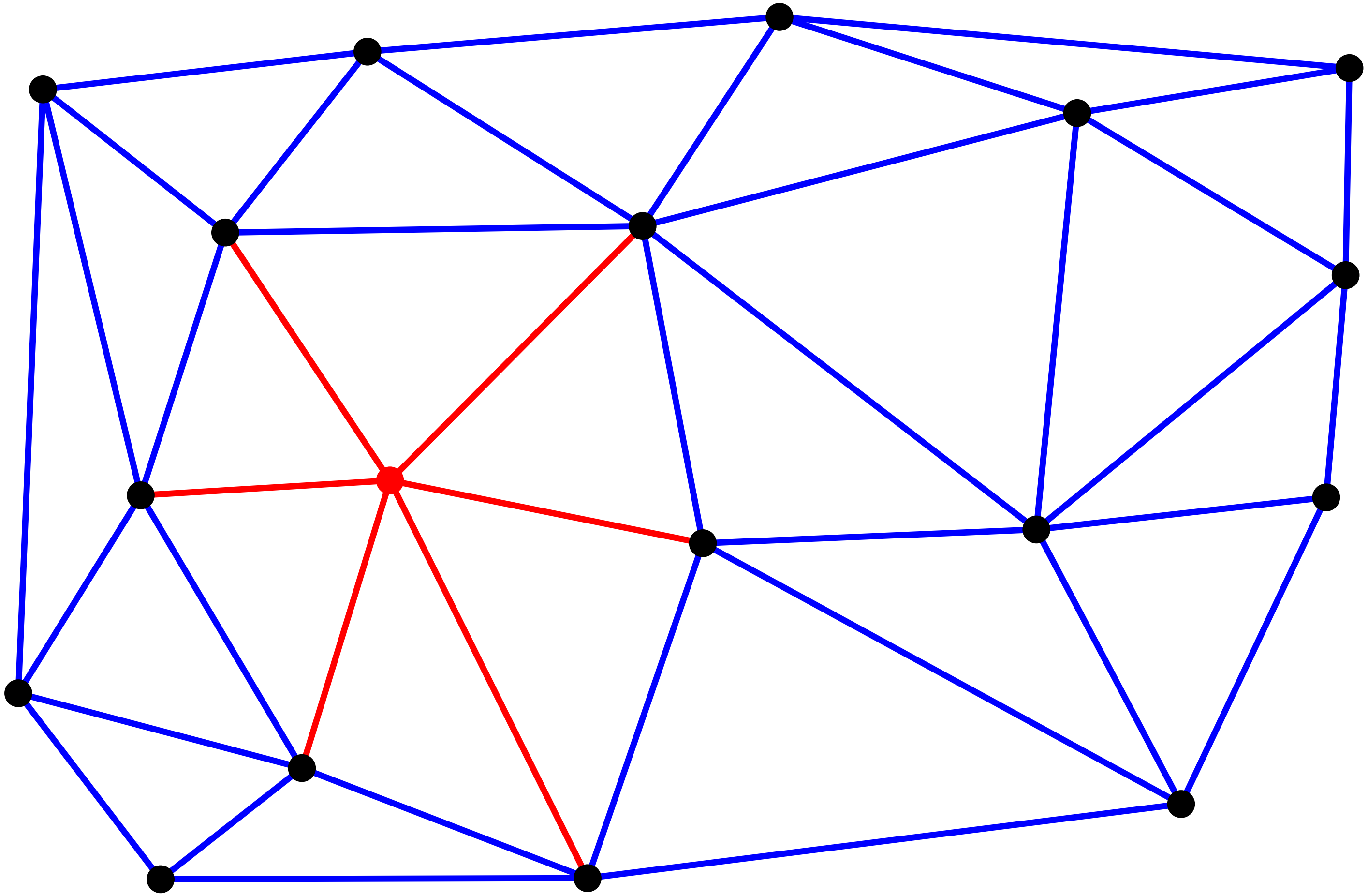
Delaunay Triangulation: incremental algorithm

New point



Delaunay Triangulation: incremental algorithm

New point



Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts

Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts $\sharp$ triangles in conflict

$\sharp$ triangles neighboring triangles in conflict

Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts $\sharp$ triangles in conflict

$\sharp$ triangles neighboring triangles in conflict

degree of new point in new triangulation

$< n$

Delaunay Triangulation: incremental algorithm

Complexity

Locate — Walk may visit all triangles

$< 2n$

Search conflicts

degree of new point in new triangulation

$< n$

Delaunay Triangulation: incremental algorithm

Complexity

Locate $O(n)$ per insertion

Search conflicts

Delaunay Triangulation: incremental algorithm

Complexity

Locate $O(n)$ per insertion

Search conflicts

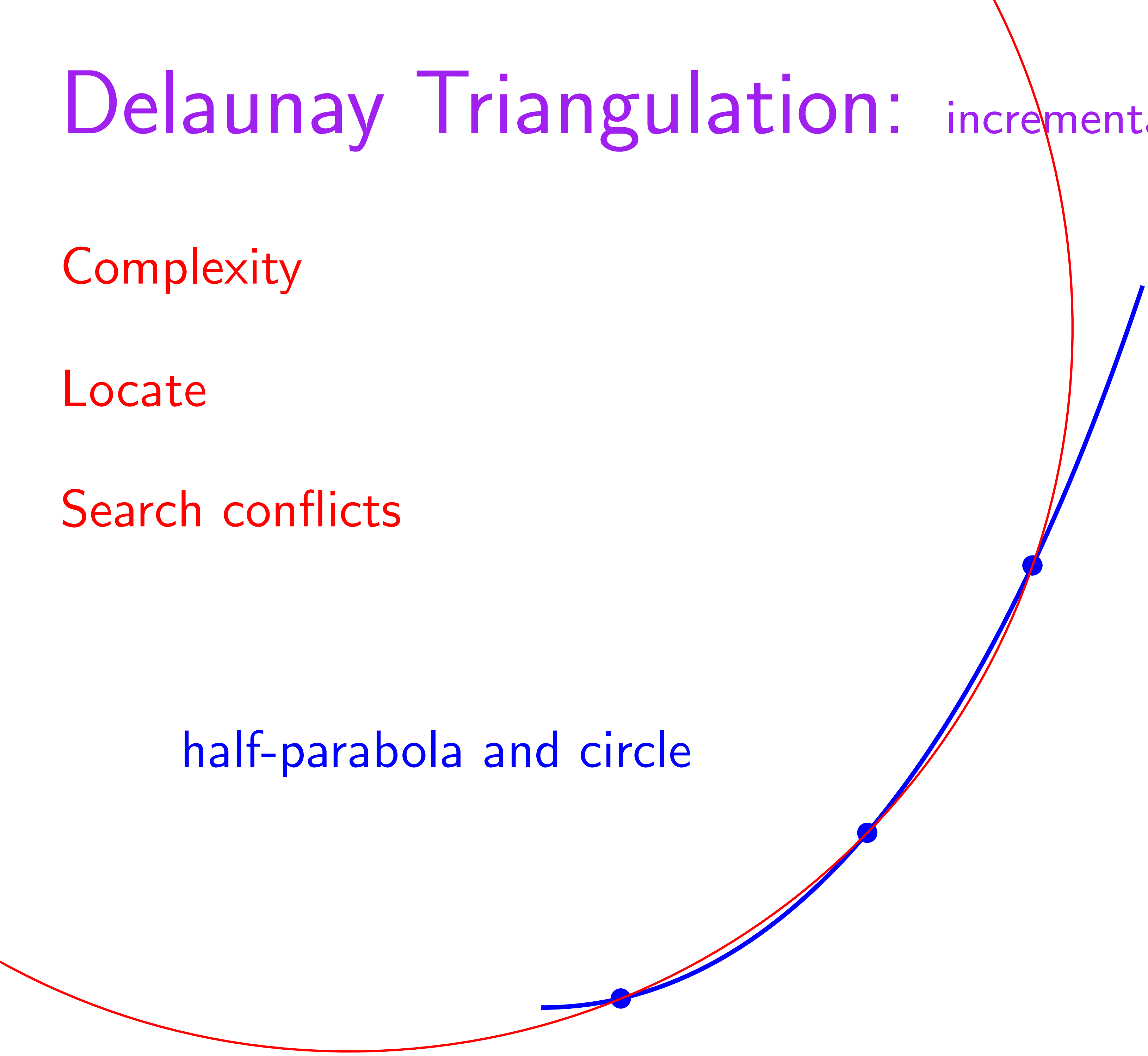
$O(n^2)$ for the whole construction

Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts

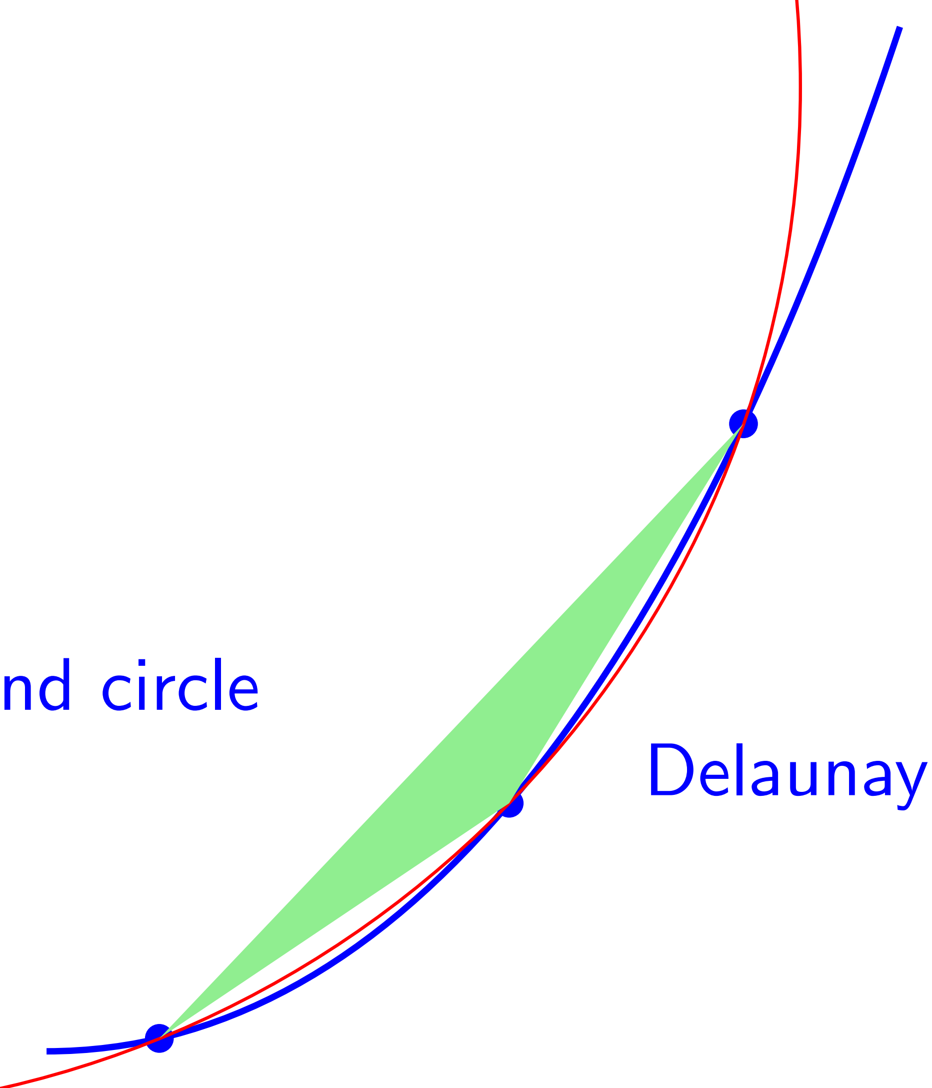


Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts

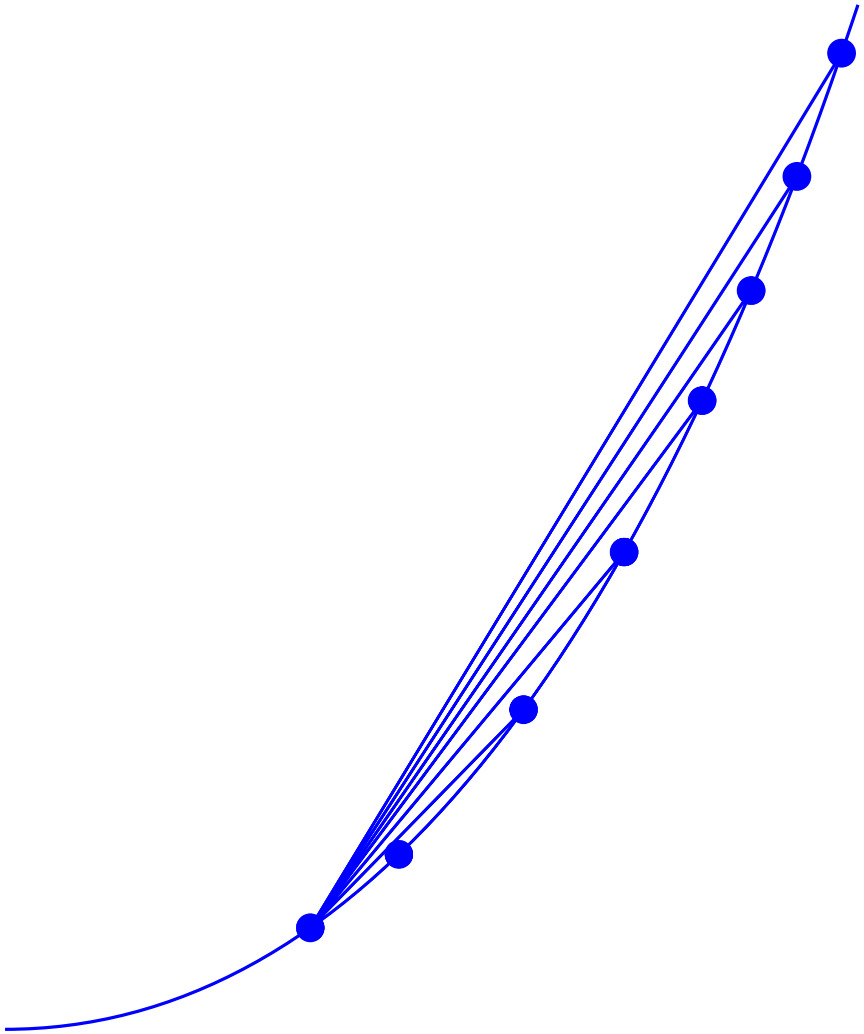


Delaunay Triangulation: incremental algorithm

Complexity

Locate

Search conflicts



Delaunay Triangulation: incremental algorithm

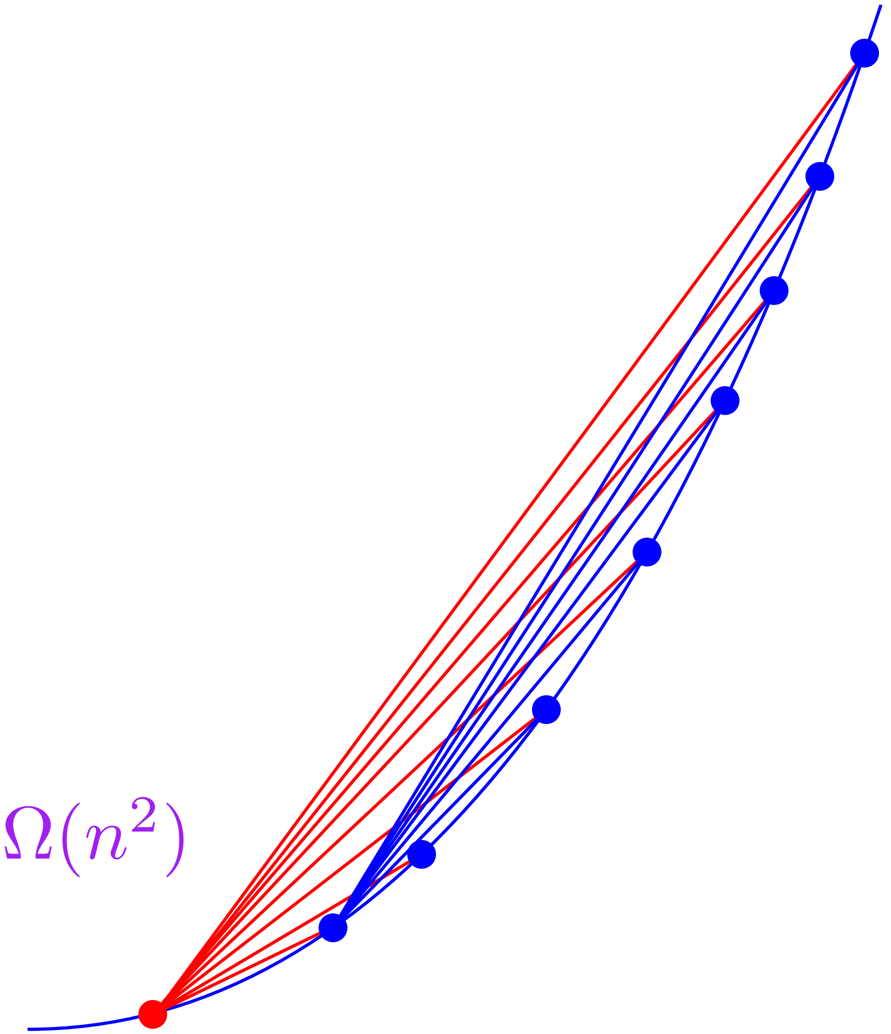
Complexity

Locate

Search conflicts

Insertion: $\Omega(n)$

Whole construction: $\Omega(n^2)$



Delaunay Triangulation: incremental algorithm

In practice

Complexity

Locate — Many possibilities (walk, Delaunay hierarchy)

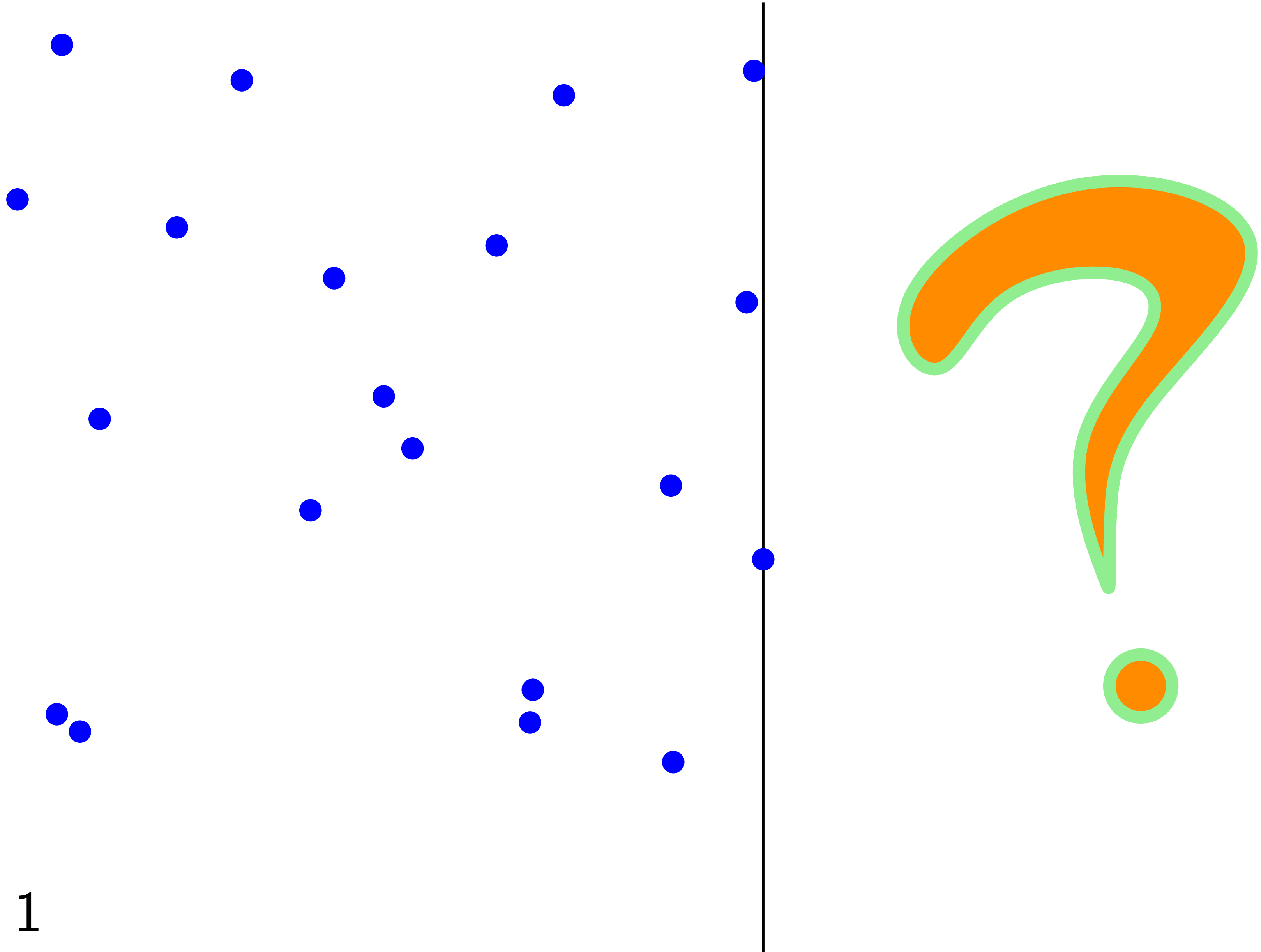
Search conflicts — Randomized

Teaser randomization lecture

Algorithm: sweep line

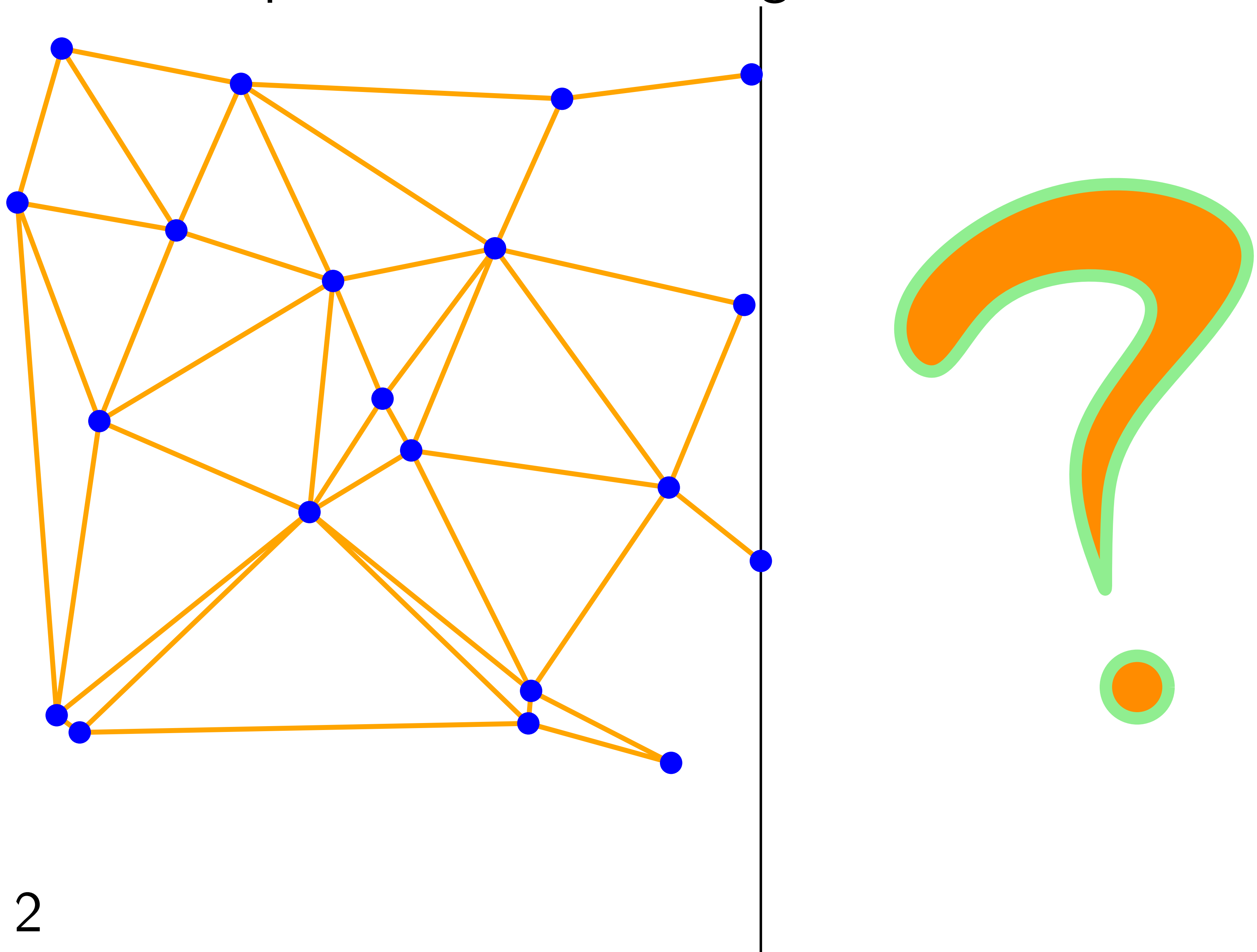
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



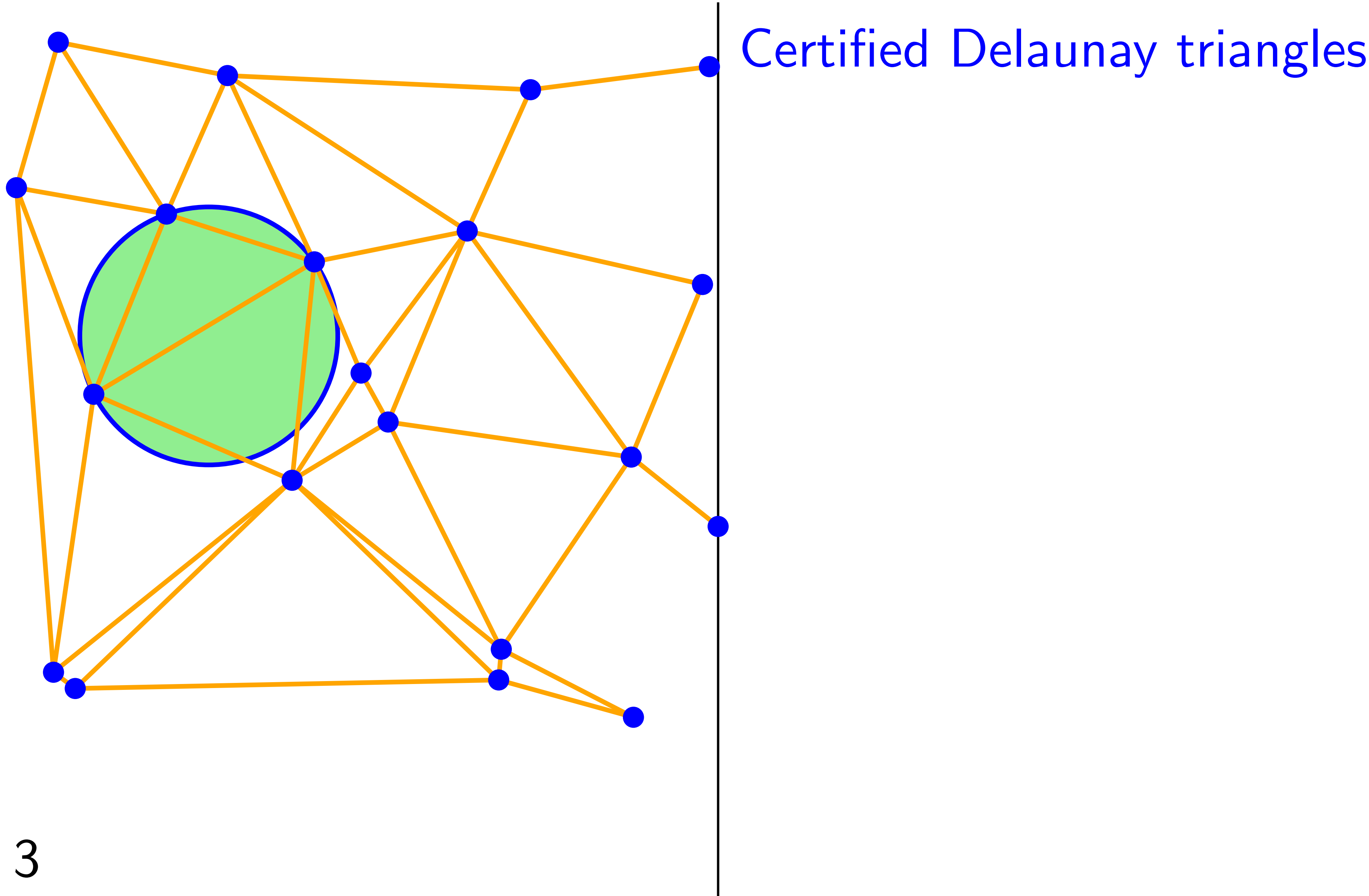
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



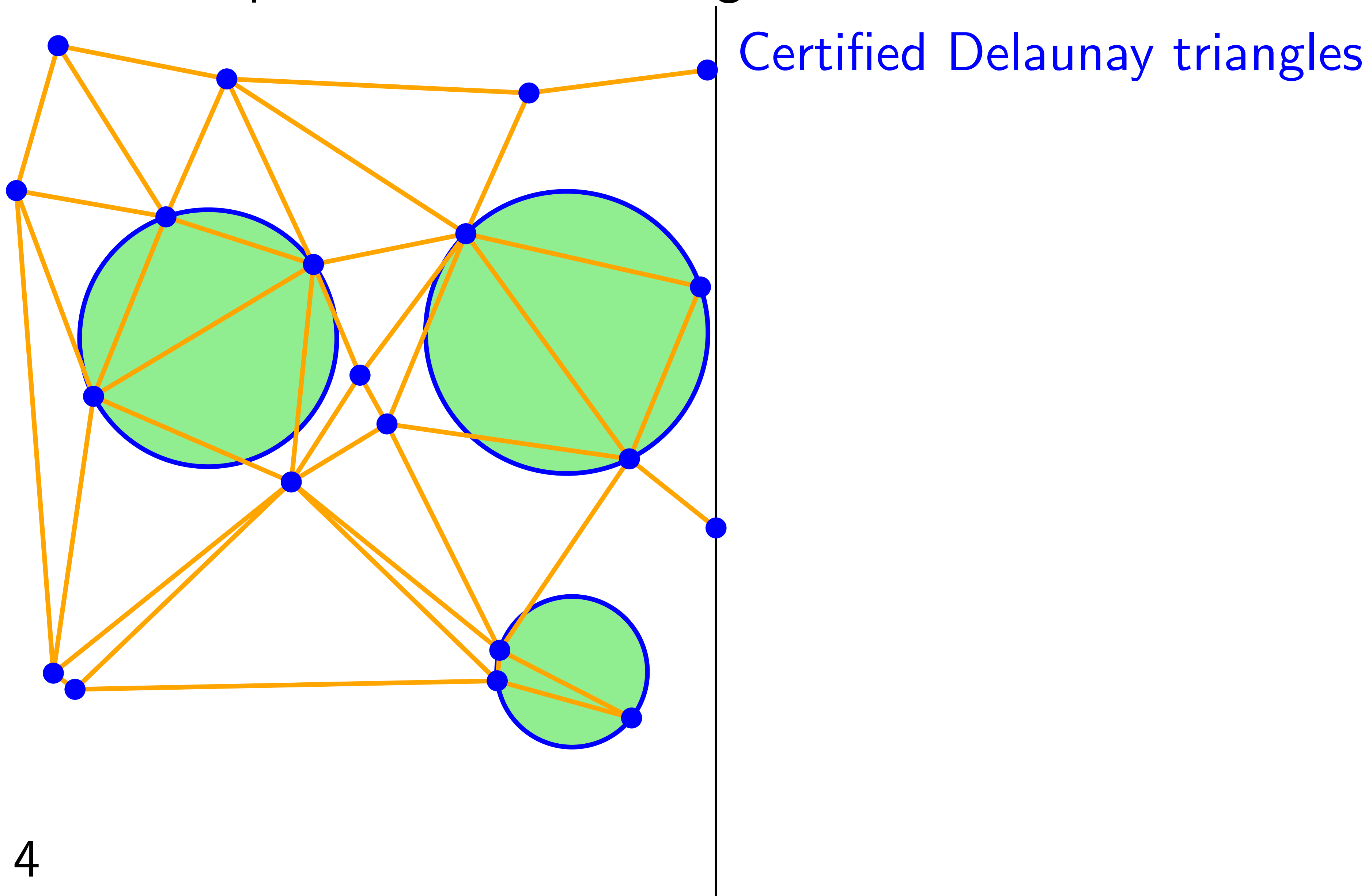
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



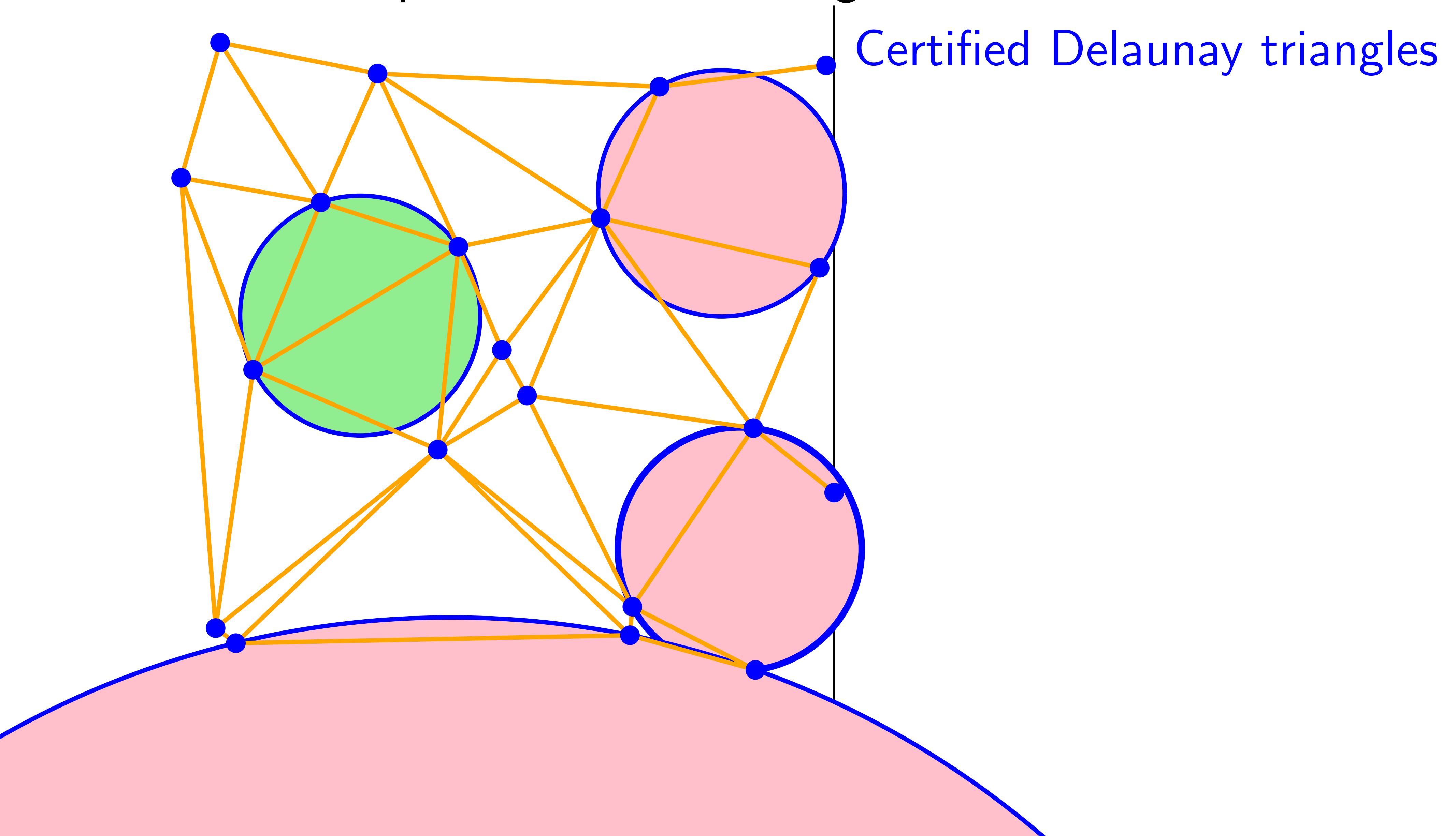
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



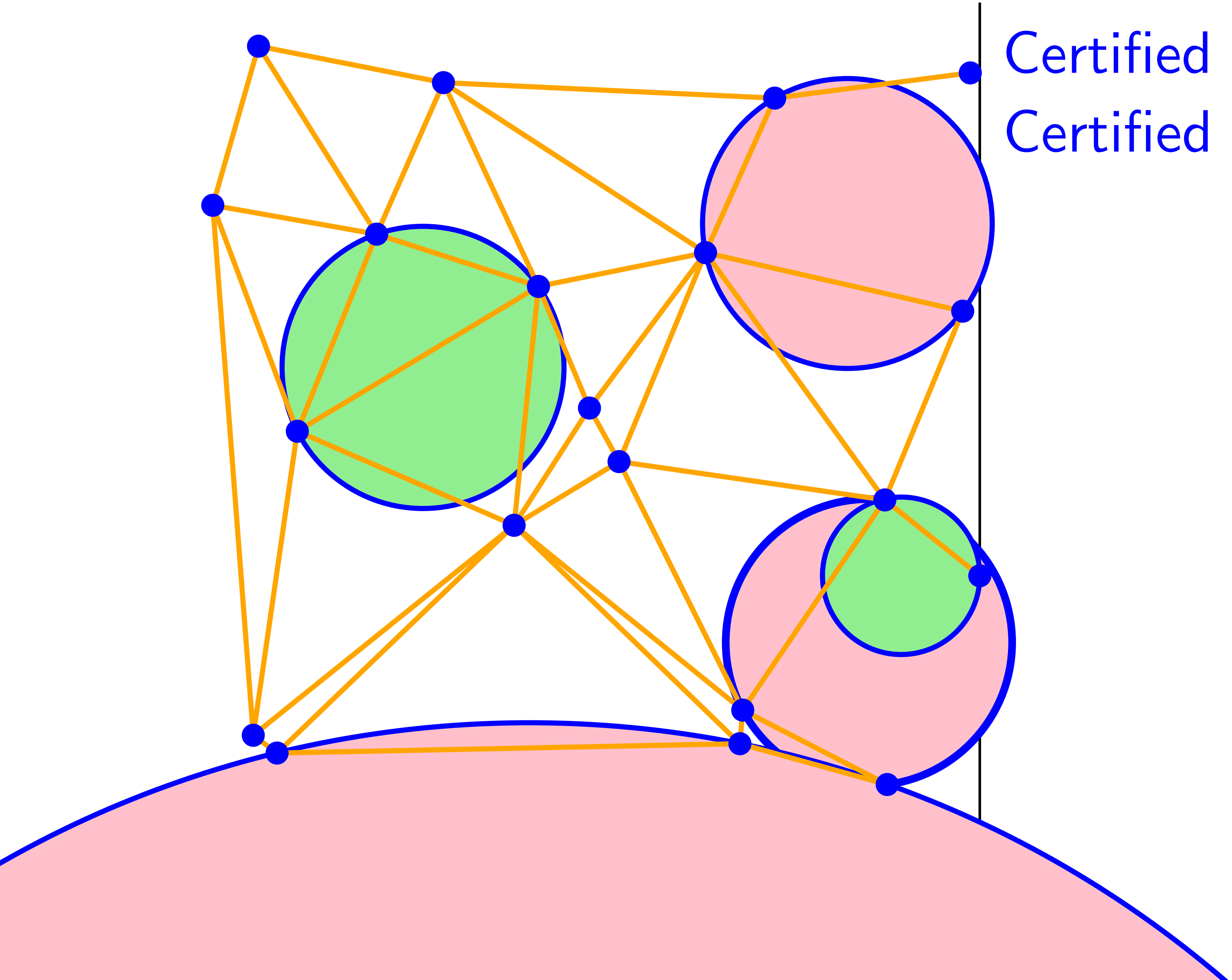
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right

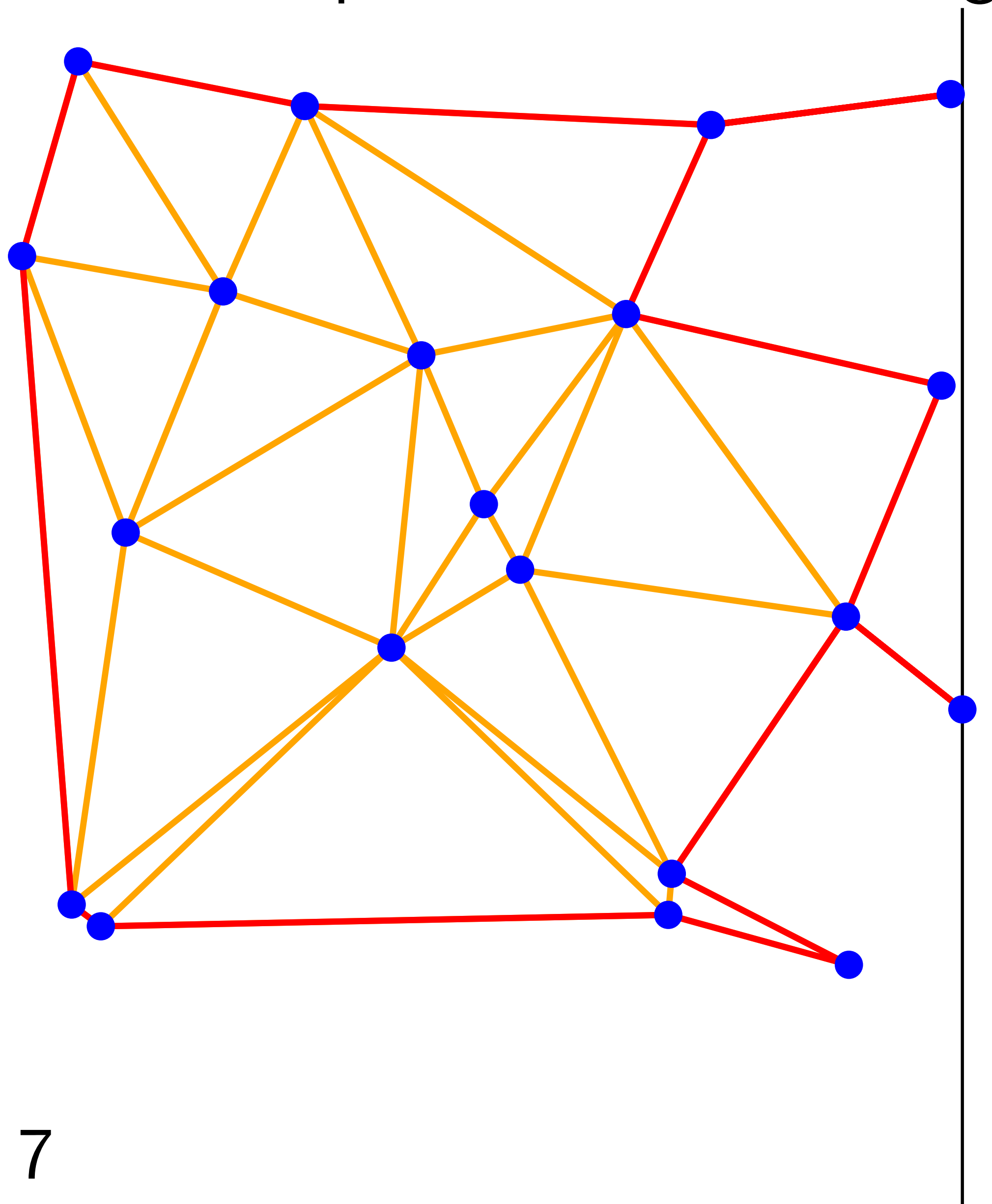


Certified Delaunay triangles

Certified Delaunay edges

Delaunay Triangulation: sweep-line algorithm

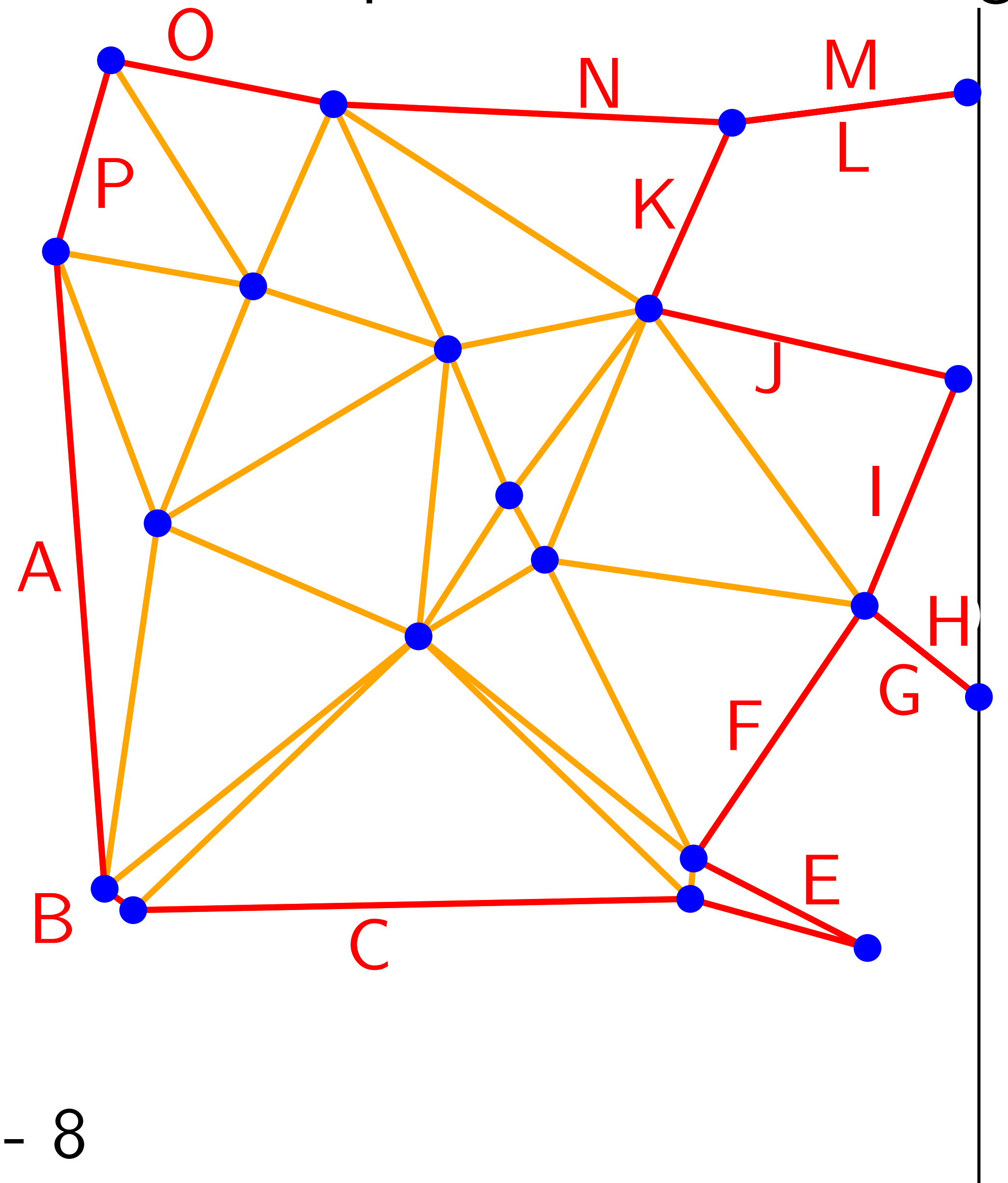
Discover the points from left to right



Boundary edges

Delaunay Triangulation: sweep-line algorithm

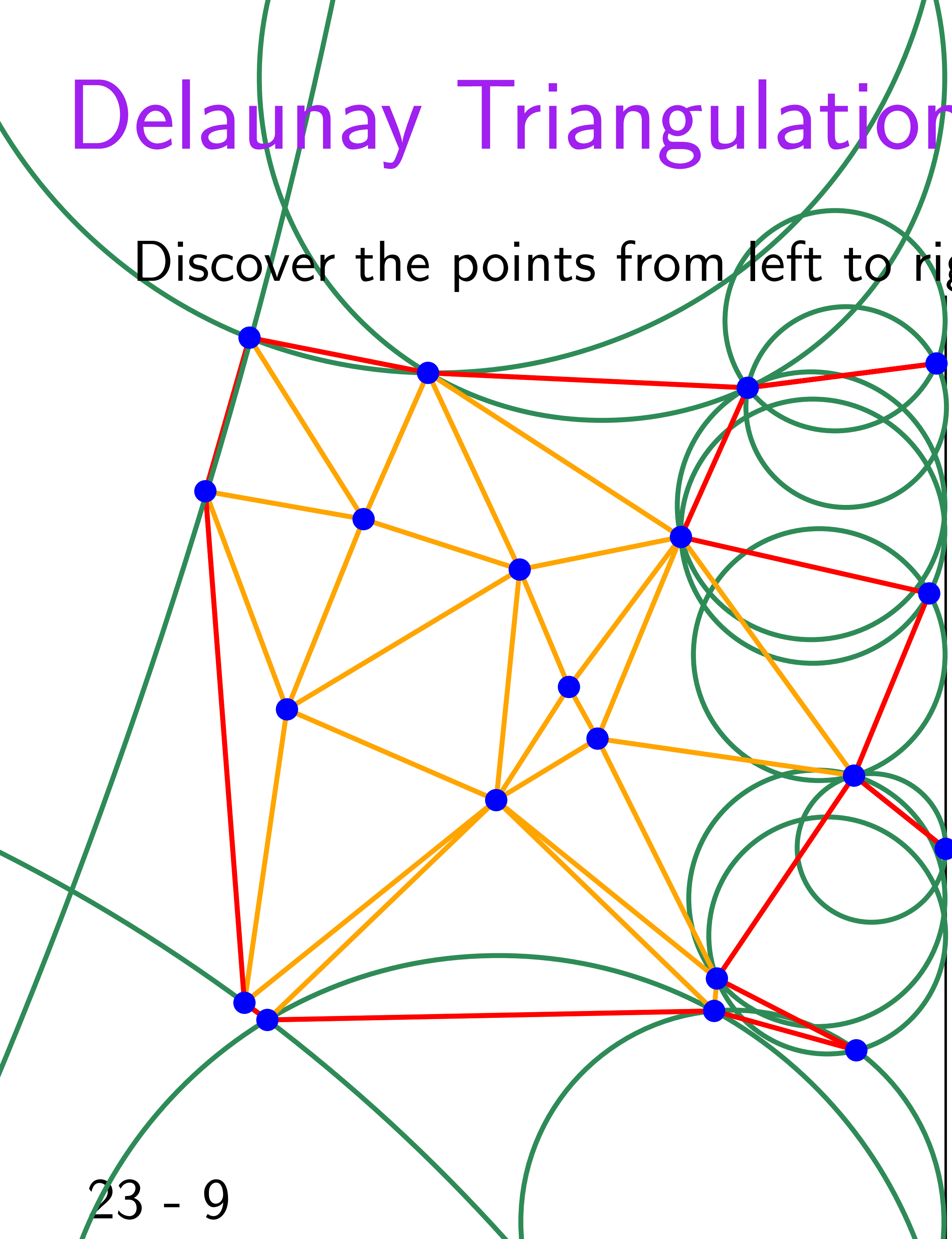
Discover the points from left to right



Boundary edges

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right

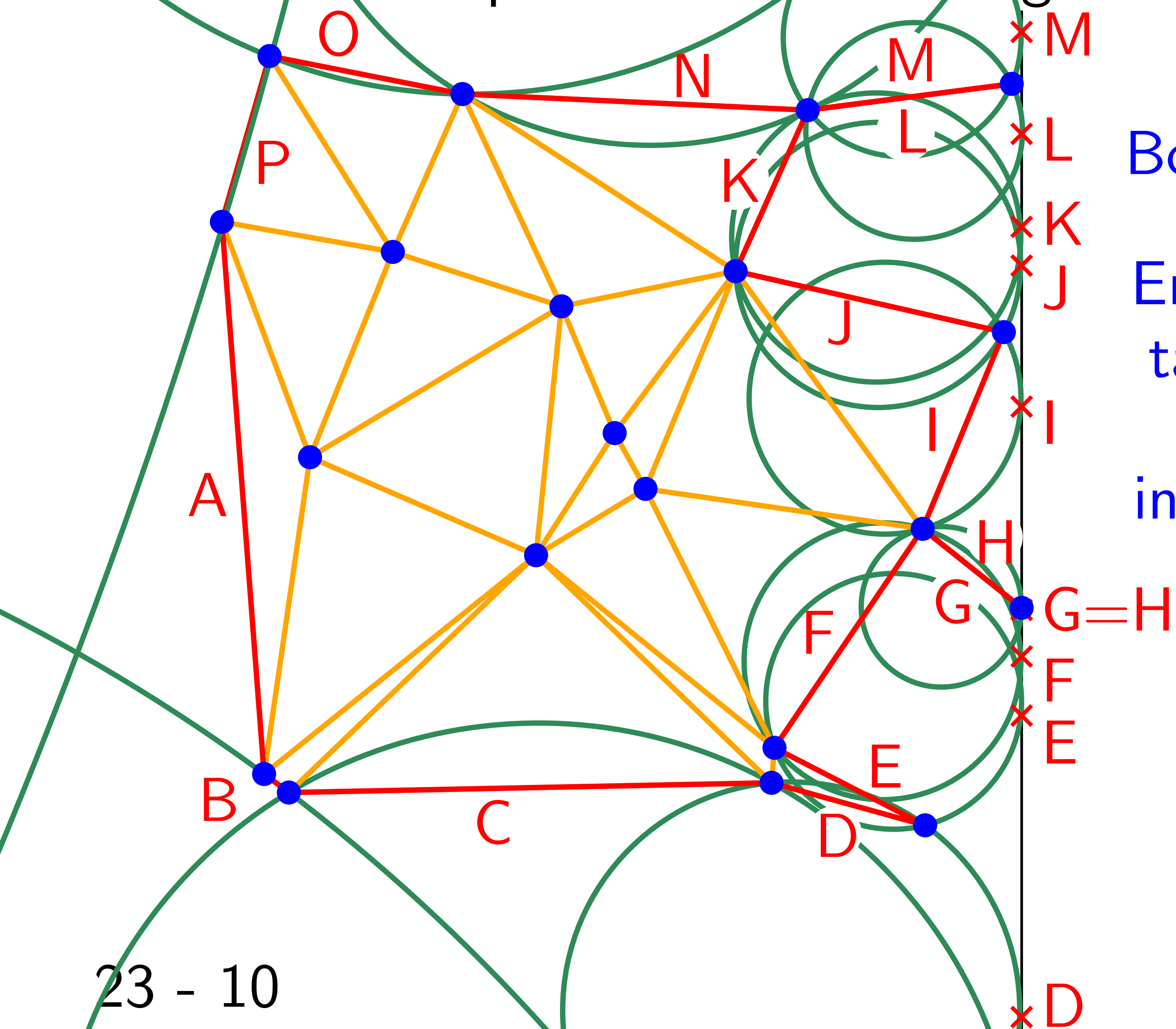


Boundary edges

Empty circles
tangent to sweep line

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



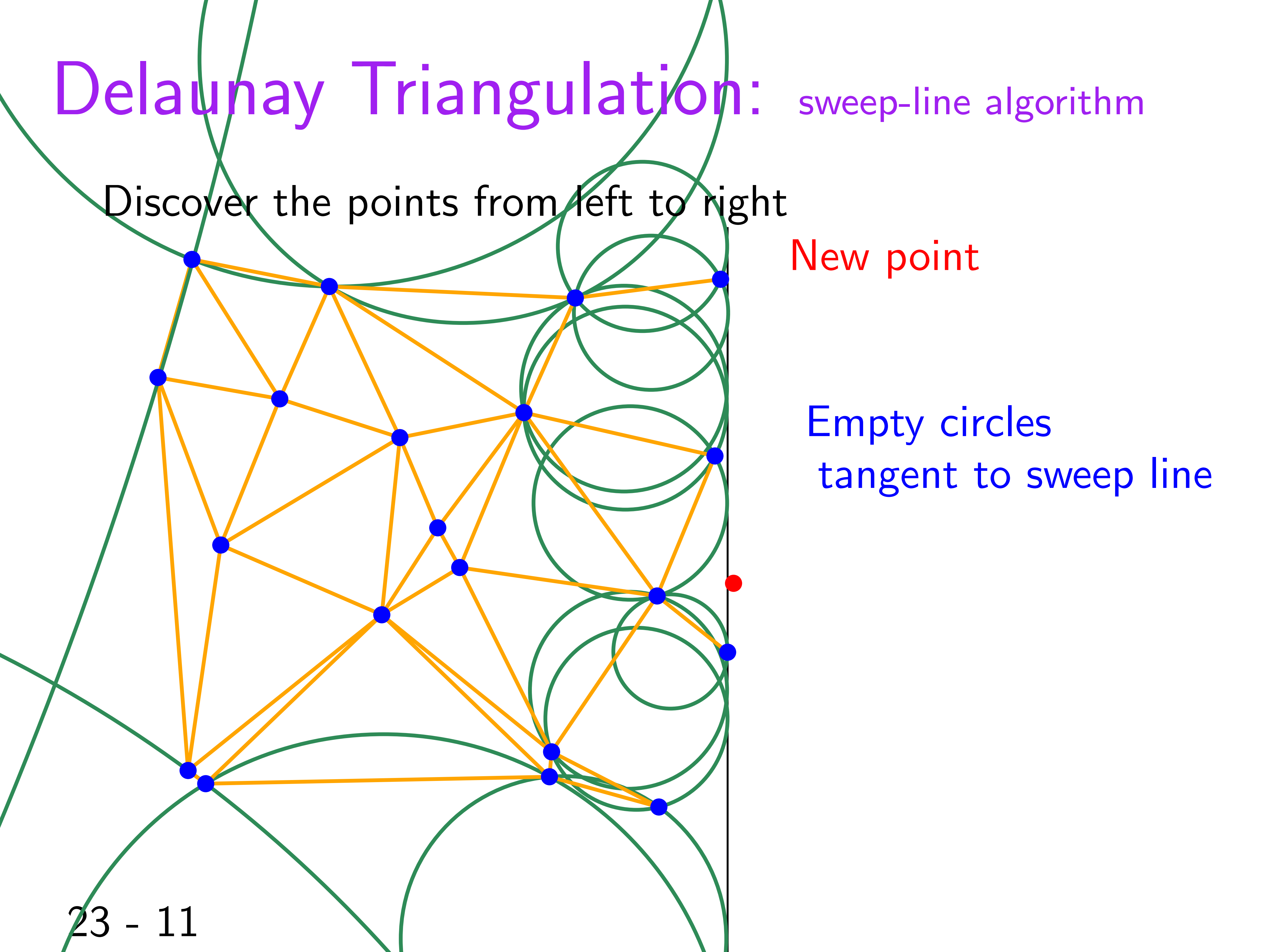
Boundary edges

Empty circles
tangent to sweep line

in order

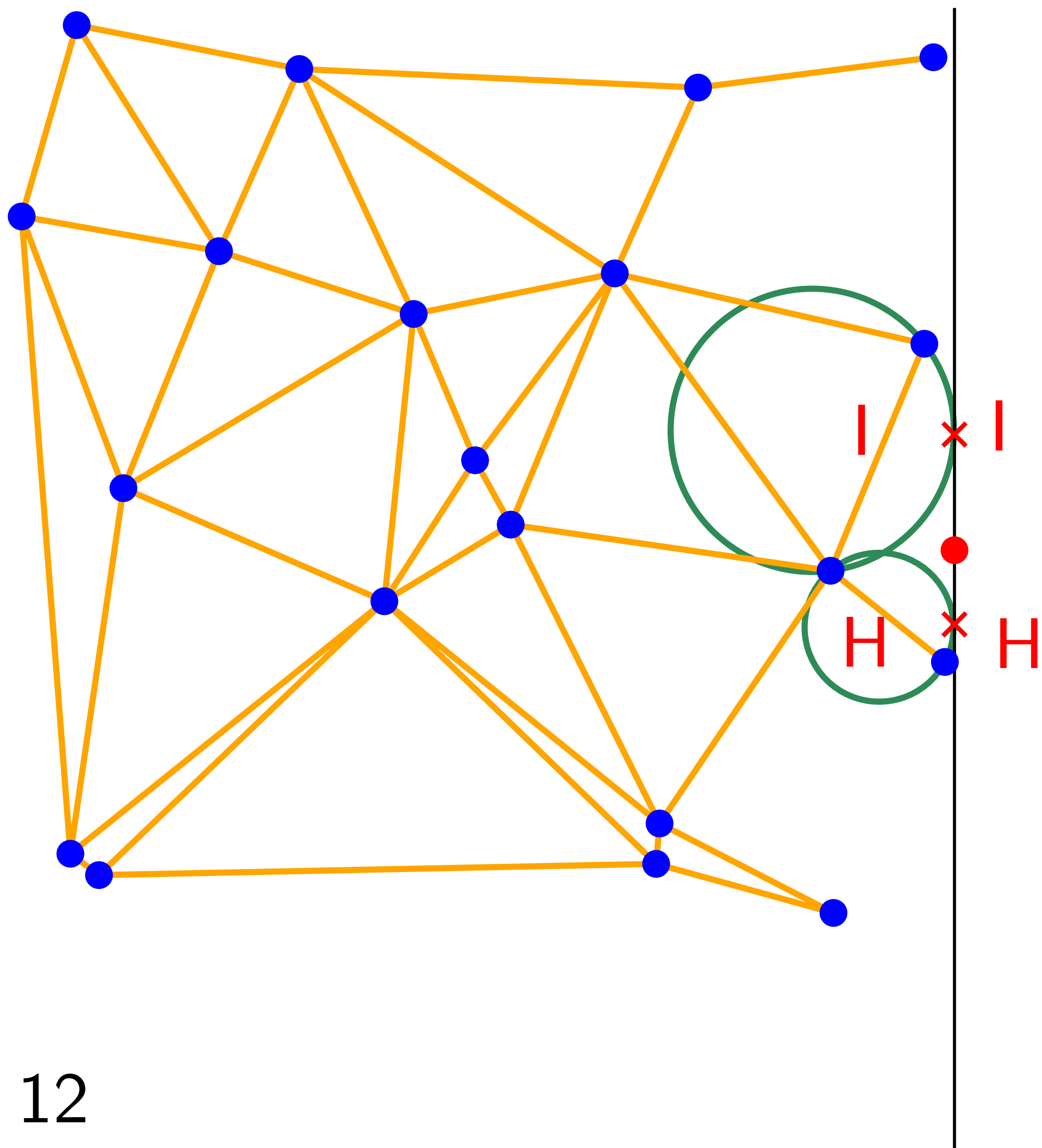
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



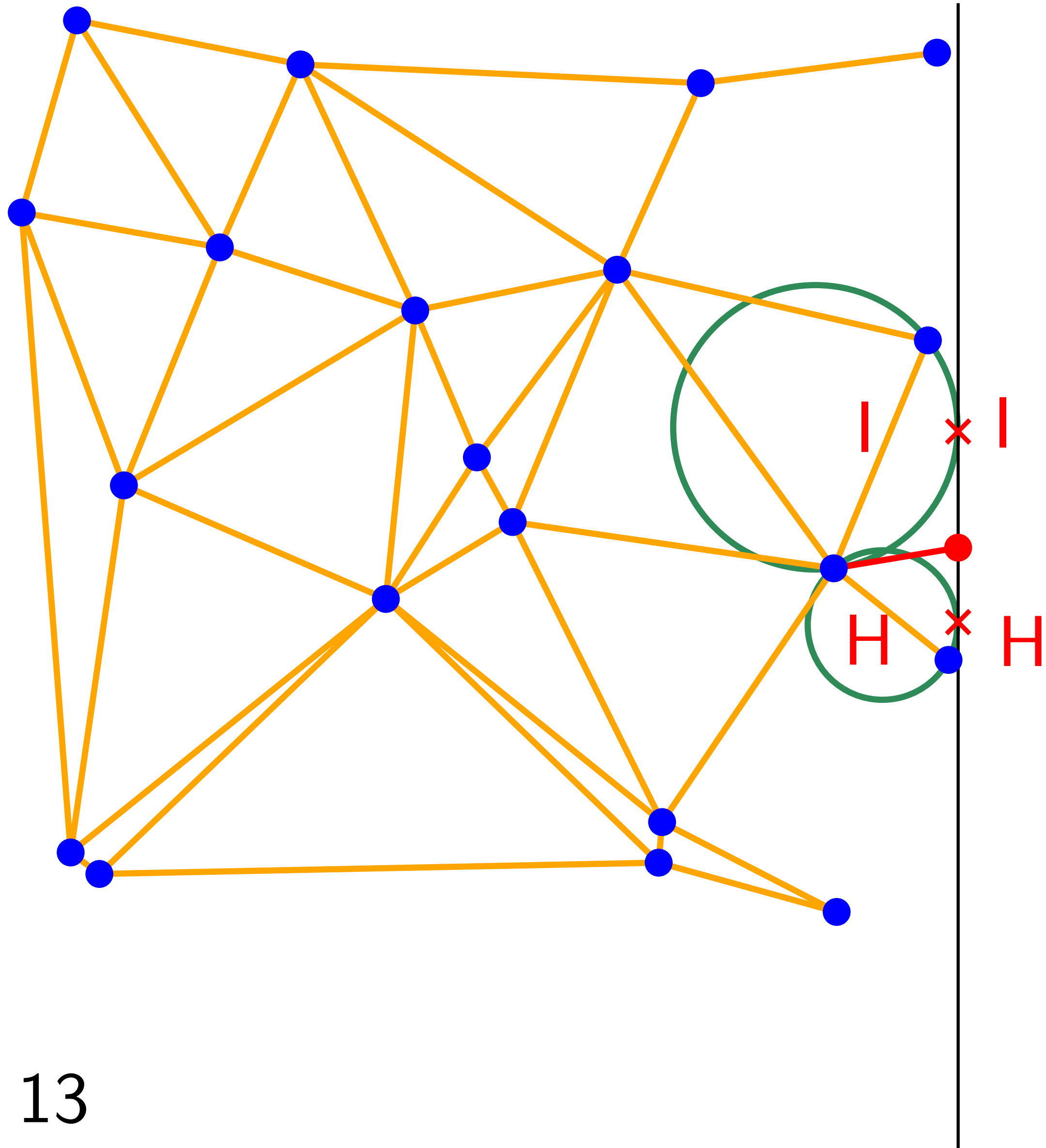
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



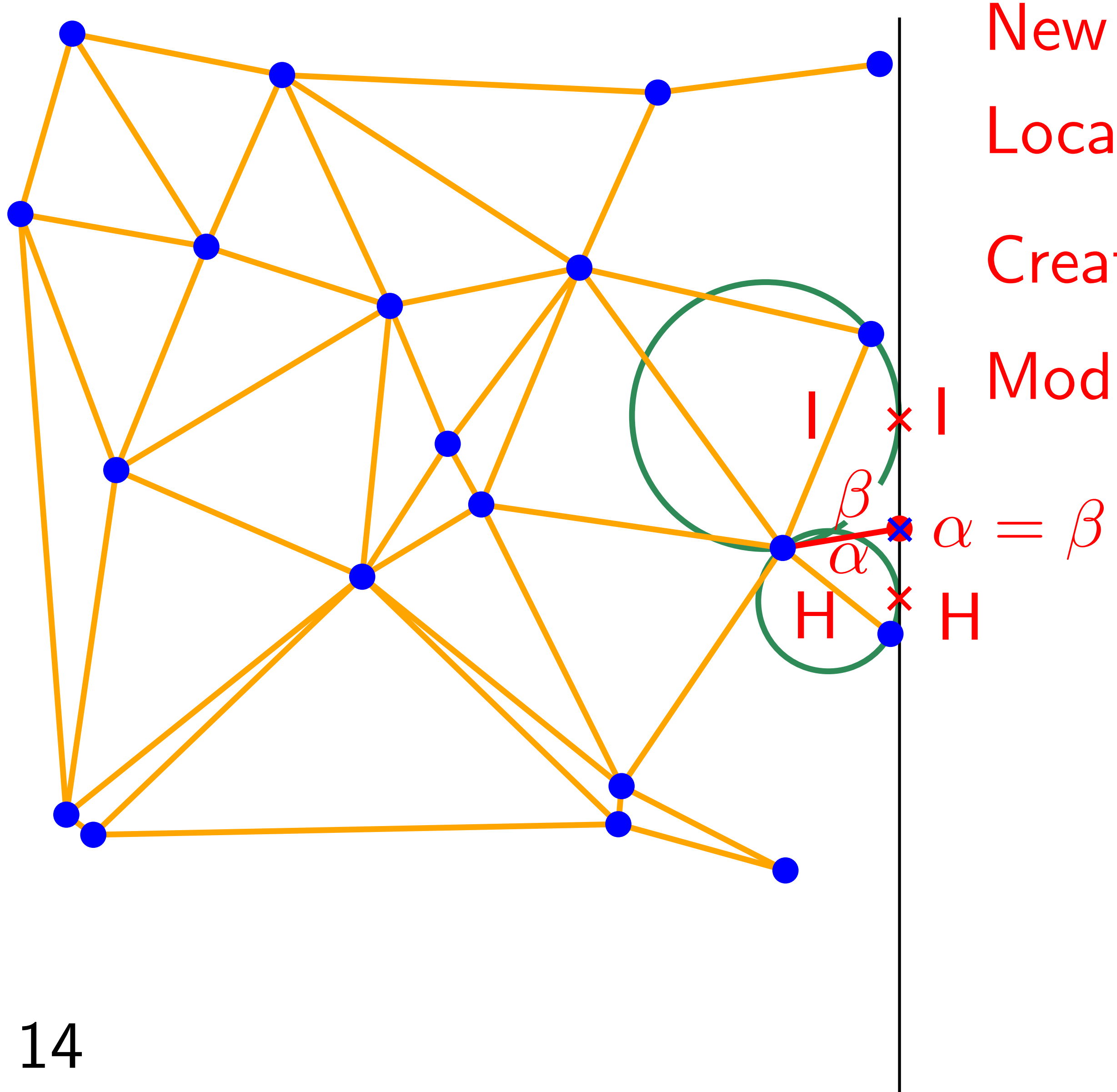
New point

Locate vertically

Create edge

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



New point

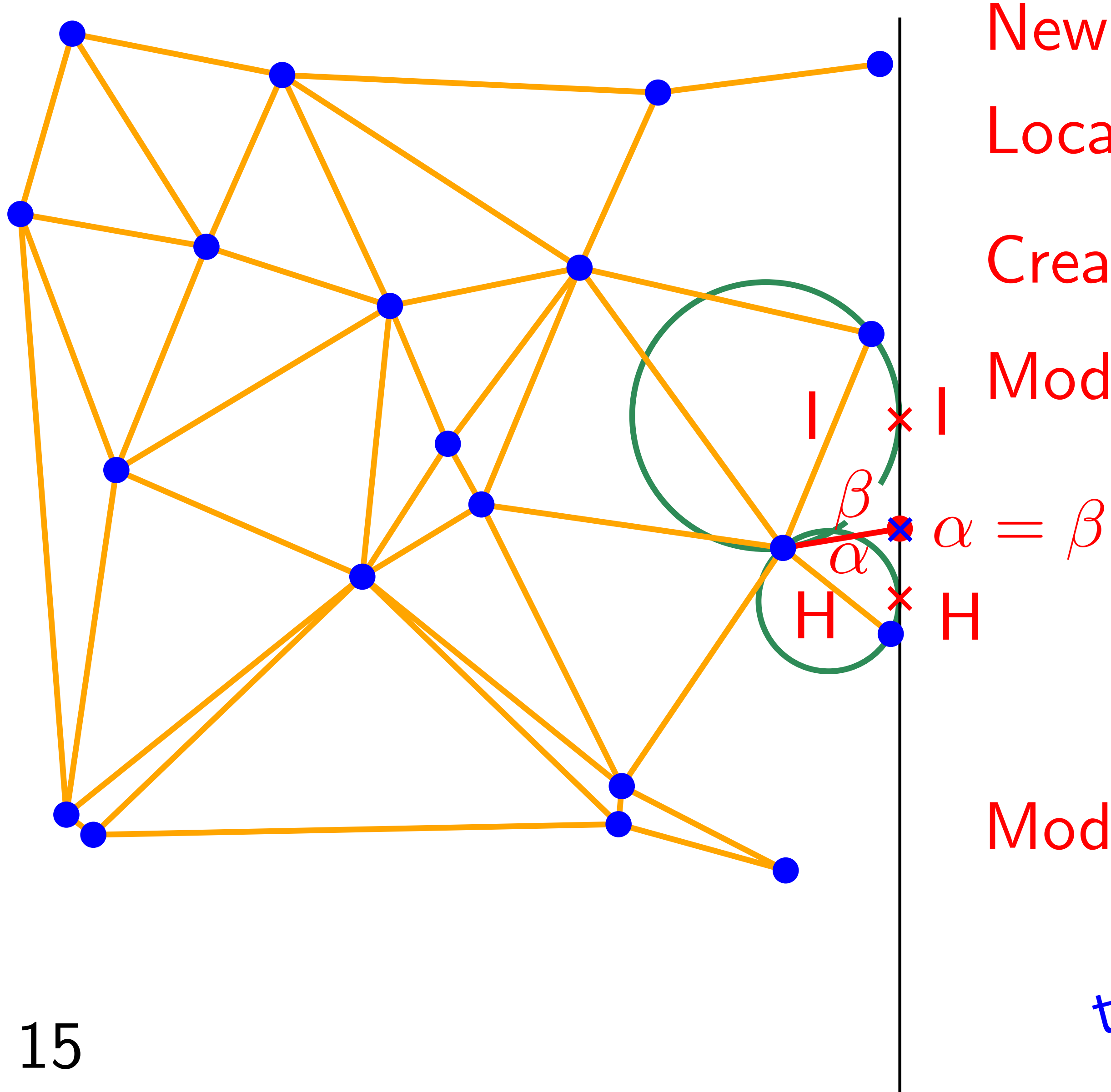
Locate vertically

Create edge

Modify boundary edges

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



New point

Locate vertically

Create edge

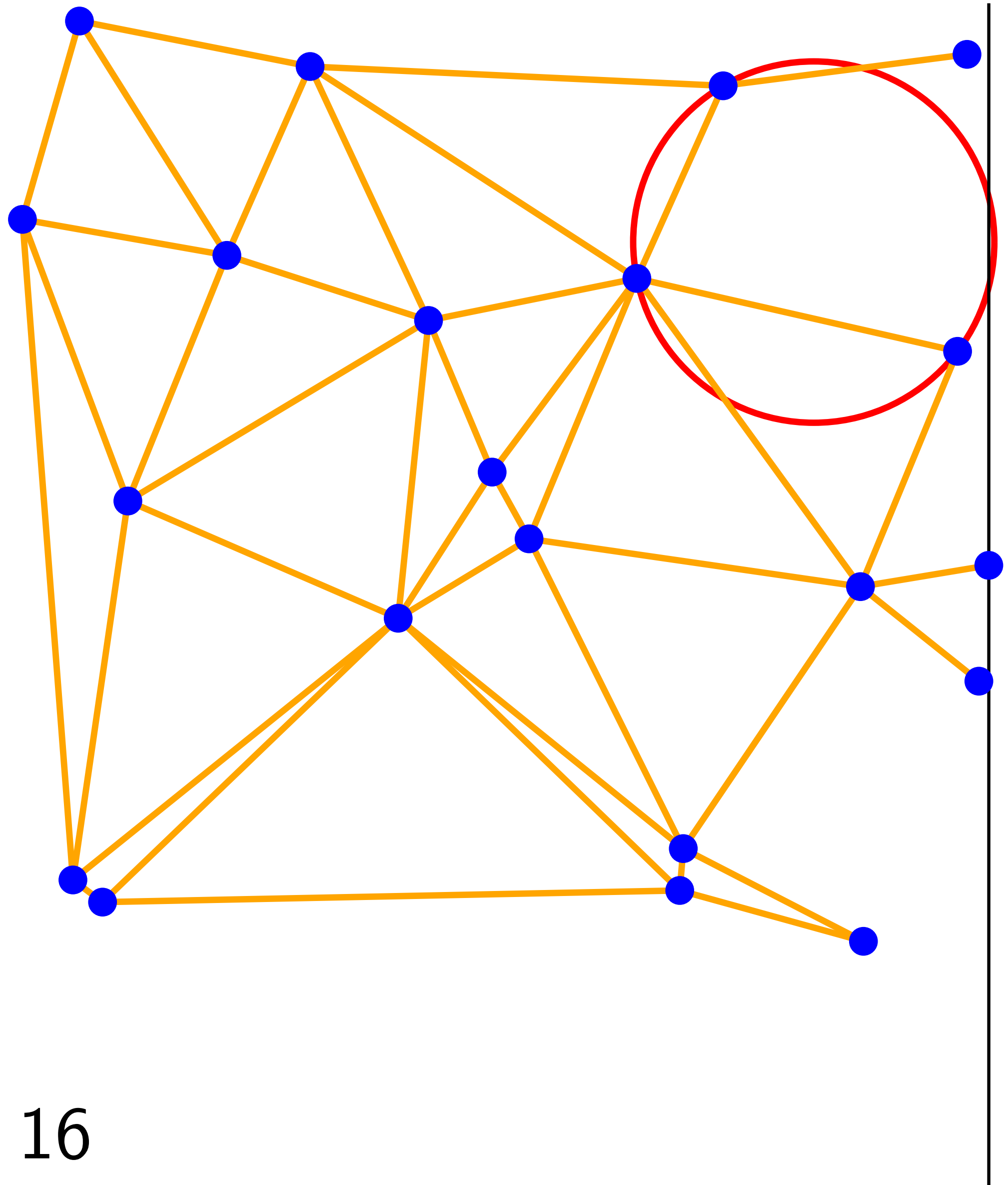
Modify boundary edges

Modify circle events

to be defined now

Delaunay Triangulation: sweep-line algorithm

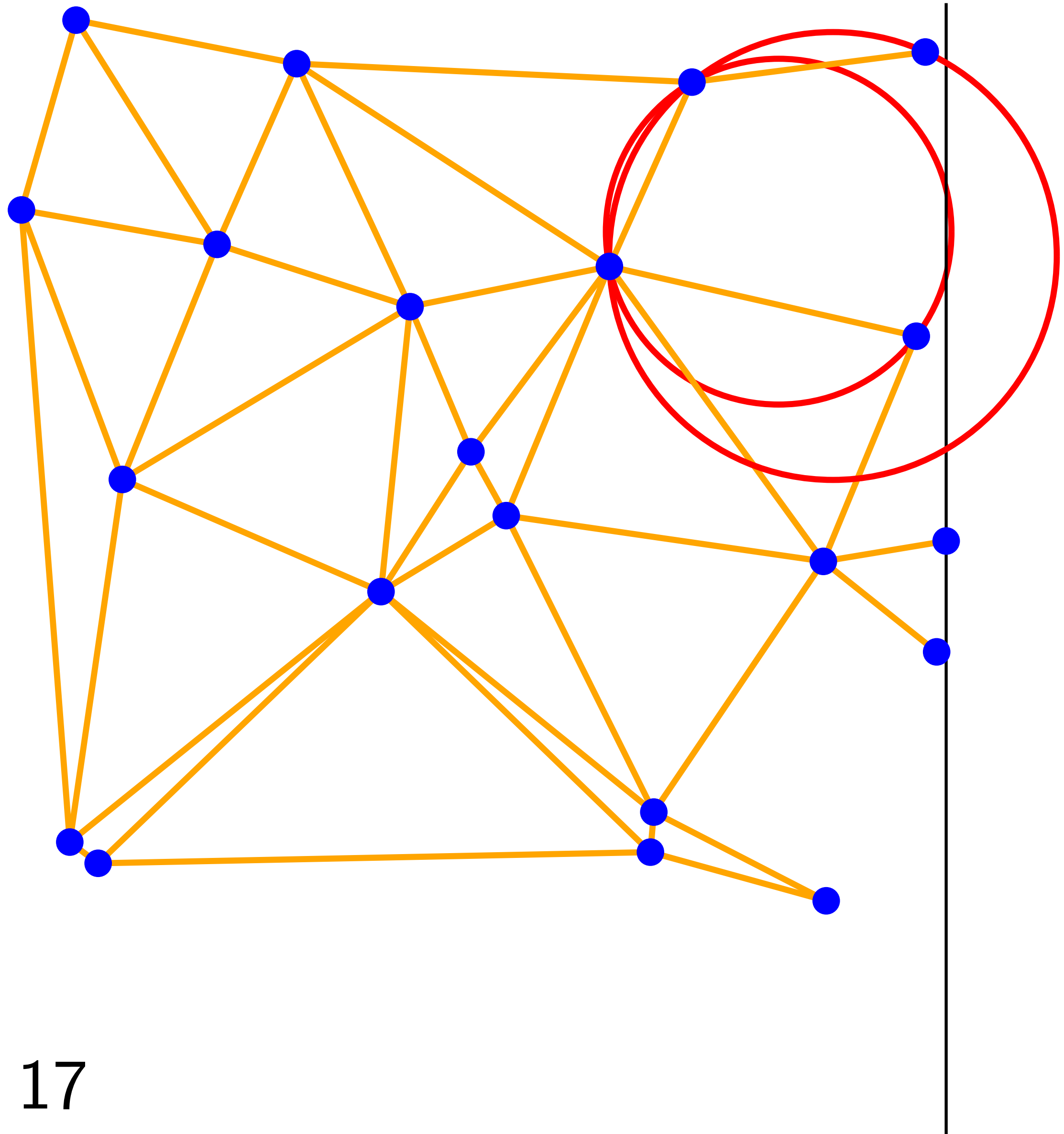
Discover the points from left to right



Closing a triangle ?

Delaunay Triangulation: sweep-line algorithm

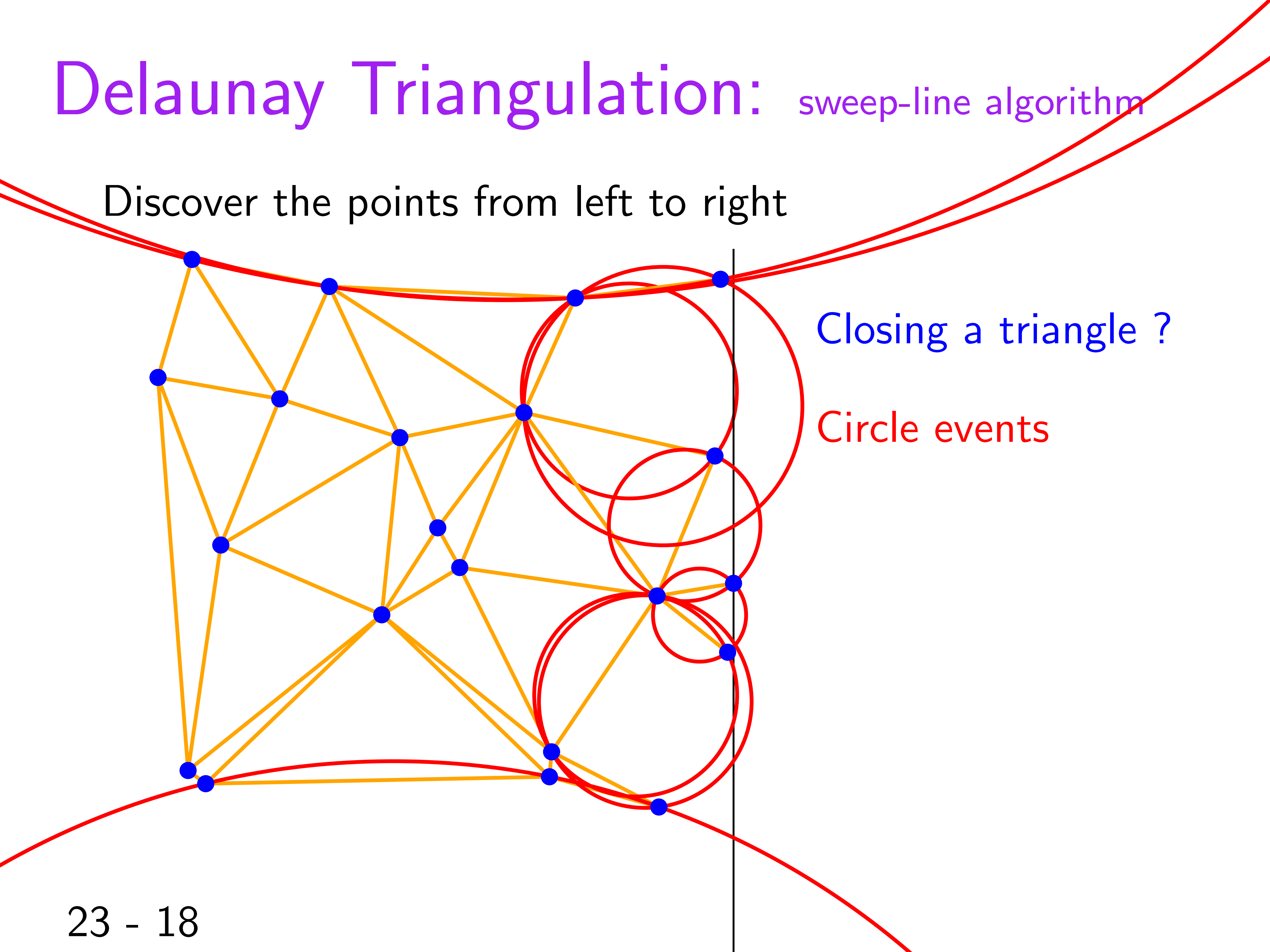
Discover the points from left to right



Closing a triangle ?

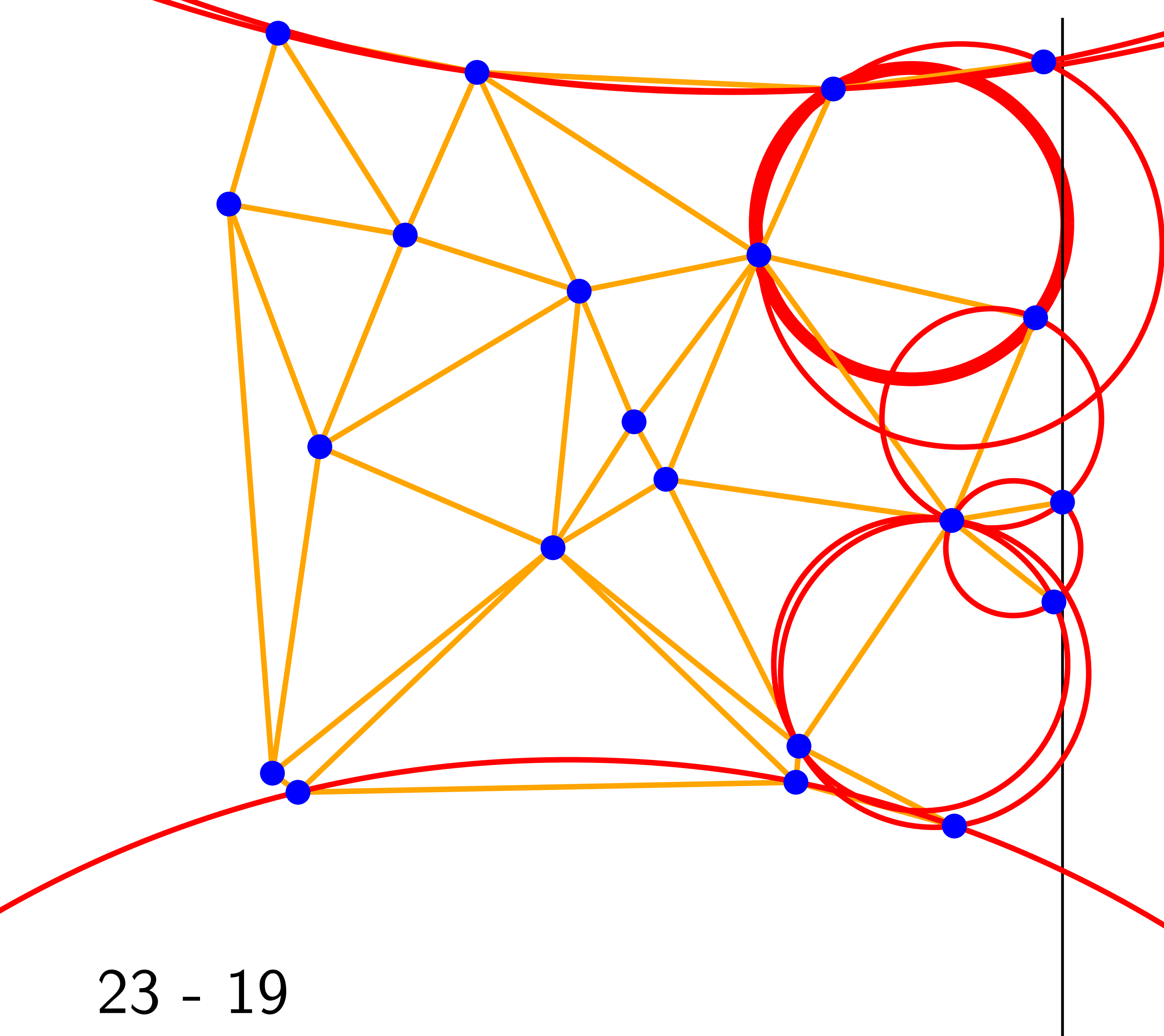
Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



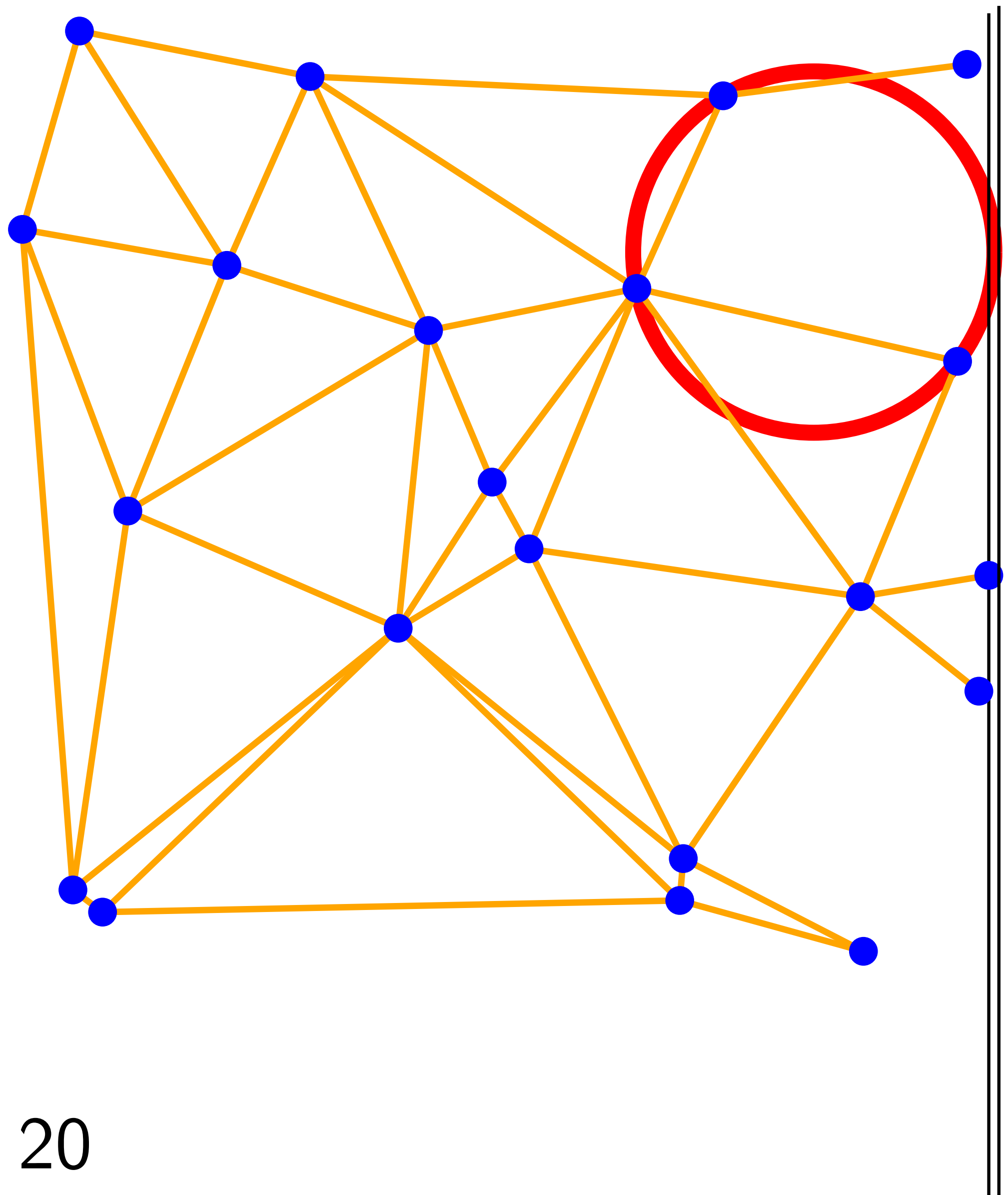
Closing a triangle ?

Circle events

Next circle event

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right

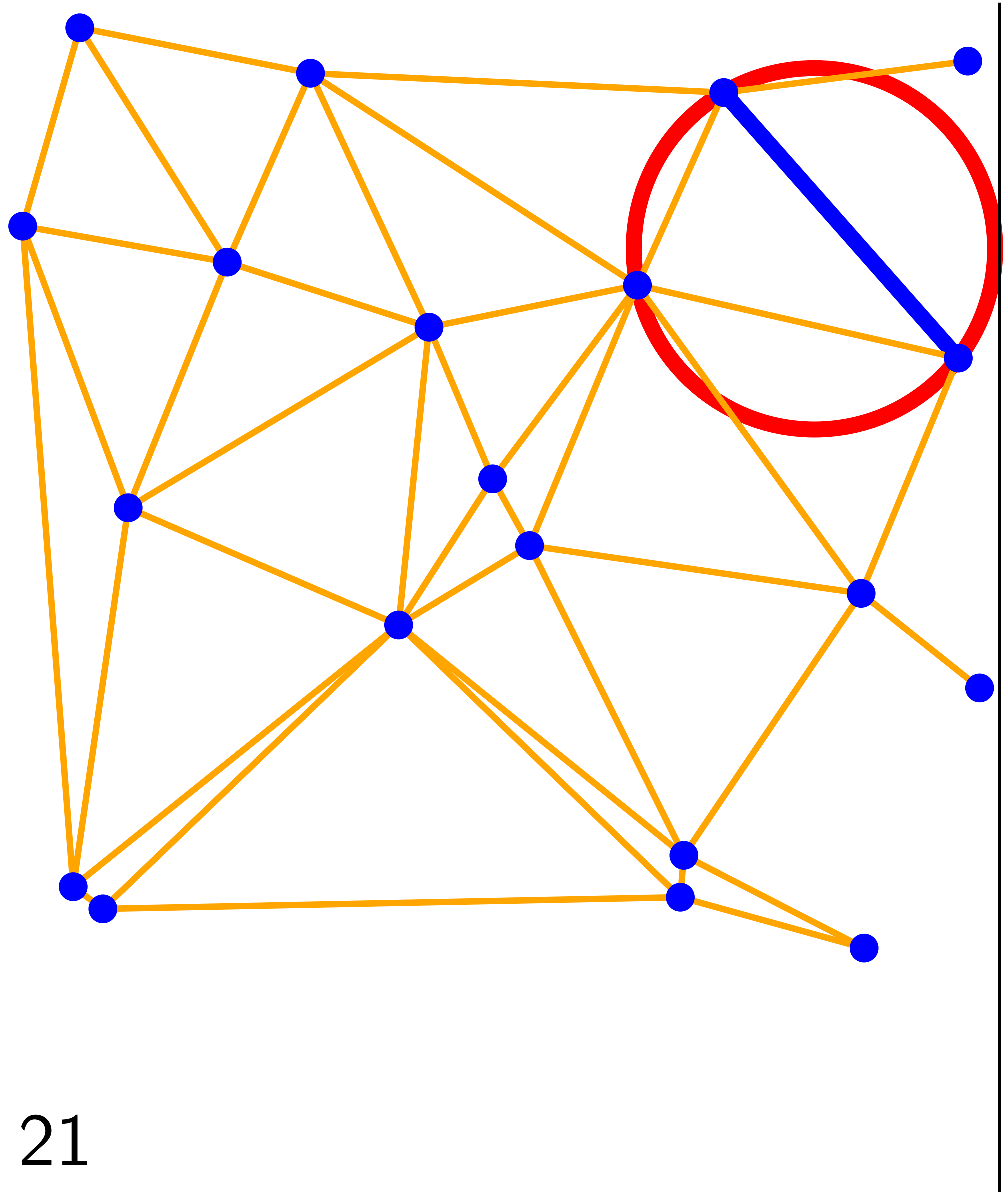


Closing a triangle ?

Next circle event

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right

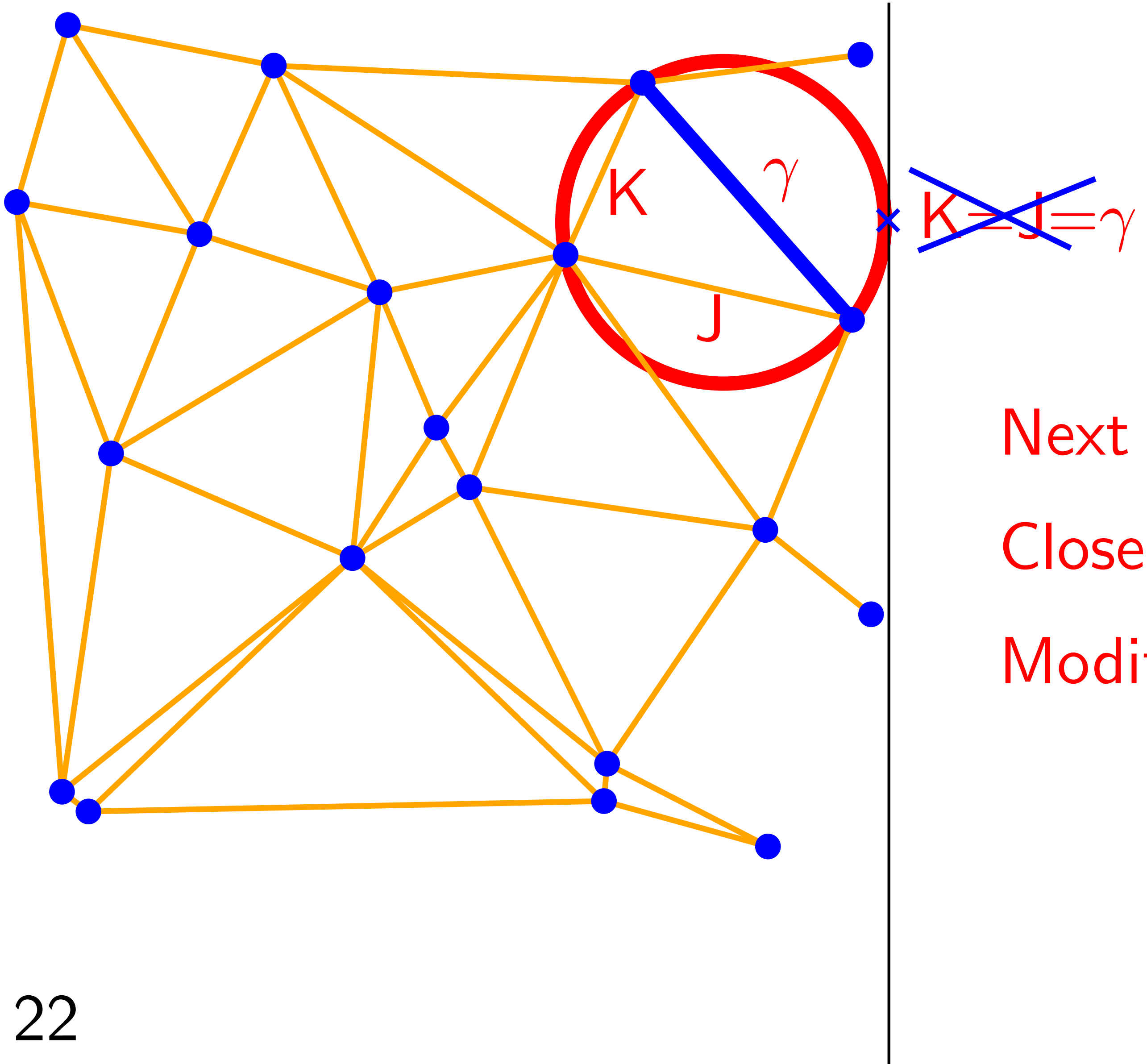


Next circle event

Close triangle

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



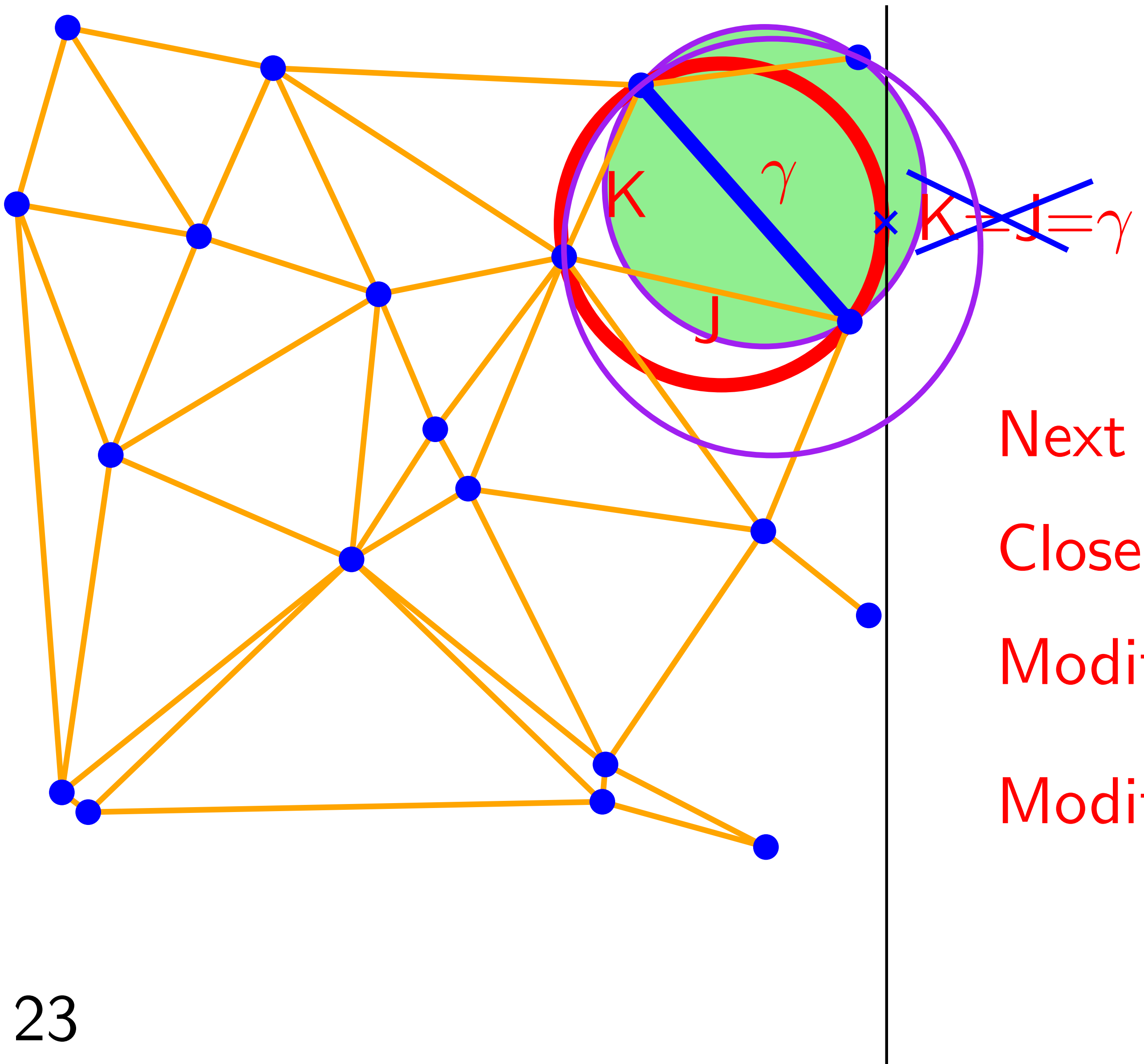
Next circle event

Close triangle

Modify boundary edges

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



Next circle event

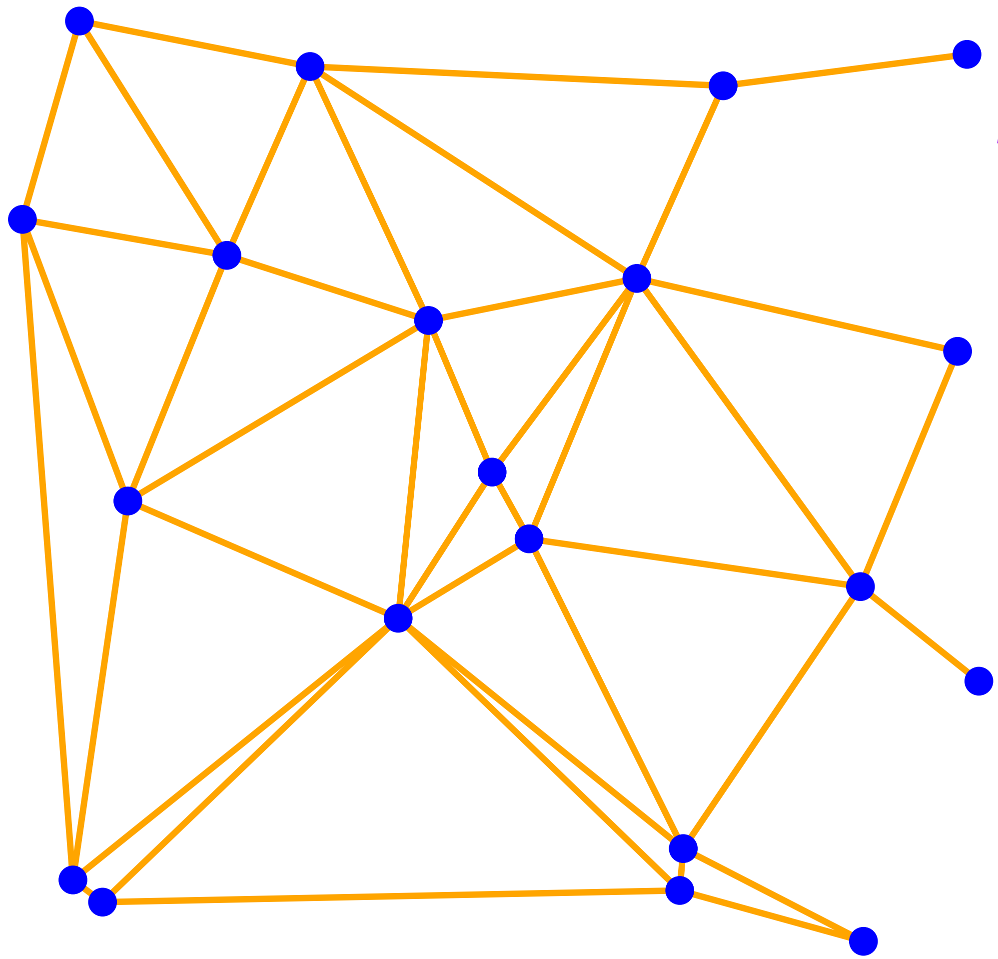
Close triangle

Modify boundary edges

Modify circle events

Delaunay Triangulation: sweep-line algorithm

Discover the points from left to right



Summary:

Process circle events
and point events
in x order

Three data structures
Triangulation
List of events (x sorted)
List of boundary edges
(ccw sorted)

Delaunay Triangulation: sweep-line algorithm

	Circle events	Point events
Complexity Number		
Triangulation		
List of events (x sorted)		
List of boundary edges (ccw sorted)		

Delaunay Triangulation: sweep-line algorithm

Complexity Number	Circle events processed	Point events
Triangulation		
List of events (x sorted)		
List of boundary edges (ccw sorted)		

Delaunay Triangulation: sweep-line algorithm

Complexity	Circle events processed	Point events
Number	$2n$	n
Triangulation		
List of events (x sorted)		
List of boundary edges (ccw sorted)		

Delaunay Triangulation: sweep-line algorithm

Complexity	Circle events processed	Point events
Number	$2n$	n
Triangulation	create 2 triangles per event	create one edge per event
List of events (x sorted)		
List of boundary edges (ccw sorted)		

Delaunay Triangulation: sweep-line algorithm

Complexity	Circle events	Point events
	processed	
Number	$2n$	n
Triangulation	create 2 triangles per event	create one edge per event
List of events (x sorted)	≤ 3 deletions ≤ 2 insertions per event	≤ 2 deletions ≤ 2 insertions per event
List of boundary edges (ccw sorted)		

Delaunay Triangulation: sweep-line algorithm

Complexity	Circle events processed	Point events
Number	$2n$	n
Triangulation	create 2 triangles per event	create one edge per event
List of events (x sorted)	≤ 3 deletions ≤ 2 insertions per event	≤ 2 deletions ≤ 2 insertions per event
List of boundary edges (ccw sorted)	replace 2 edges by 1 per event	locate, then insert 2 edges per event

Delaunay Triangulation: sweep-line algorithm

Complexity Number	Circle events processed $2n$	Point events n
Triangulation $O(1)$ per operation	create 2 triangles per event	create one edge per event
List of events (x sorted) $O(\log n)$ per operation	≤ 3 deletions ≤ 2 insertions per event	≤ 2 deletions ≤ 2 insertions per event
List of boundary edges (ccw sorted) $O(\log n)$ per operation	replace 2 edges by 1 per event	locate, then insert 2 edges per event

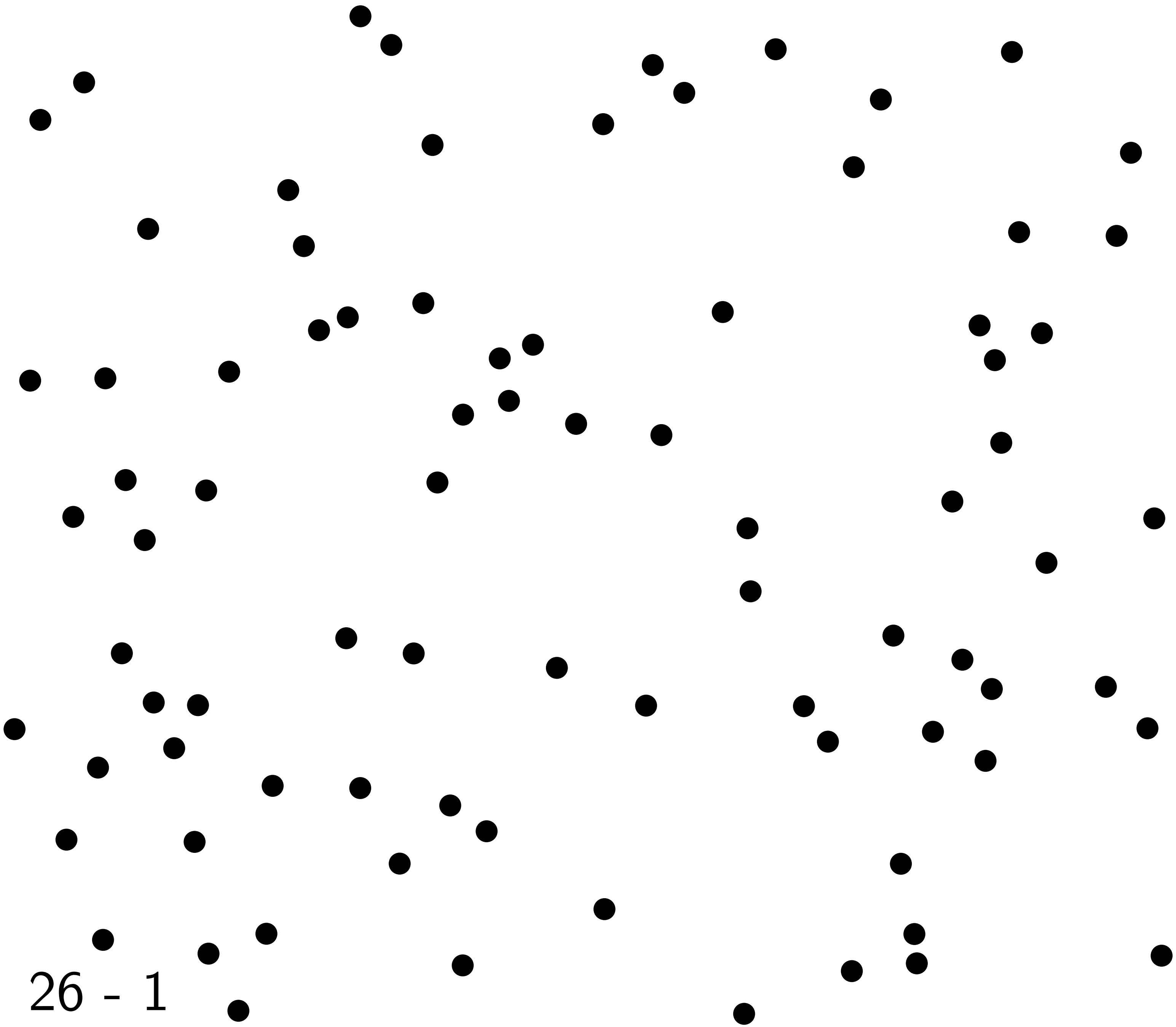
Delaunay Triangulation: sweep-line algorithm

Complexity Number	Circle events processed $2n$	Point events n
Triangulation $O(1)$ per operation	create 2 triangles	create one edge
List of events (x sorted) $O(\log n)$ per operation	per event	per event
List of boundary edges (ccw sorted) $O(\log n)$ per operation	replace 2 edges by 1 per event	locate, then insert 2 edges per event

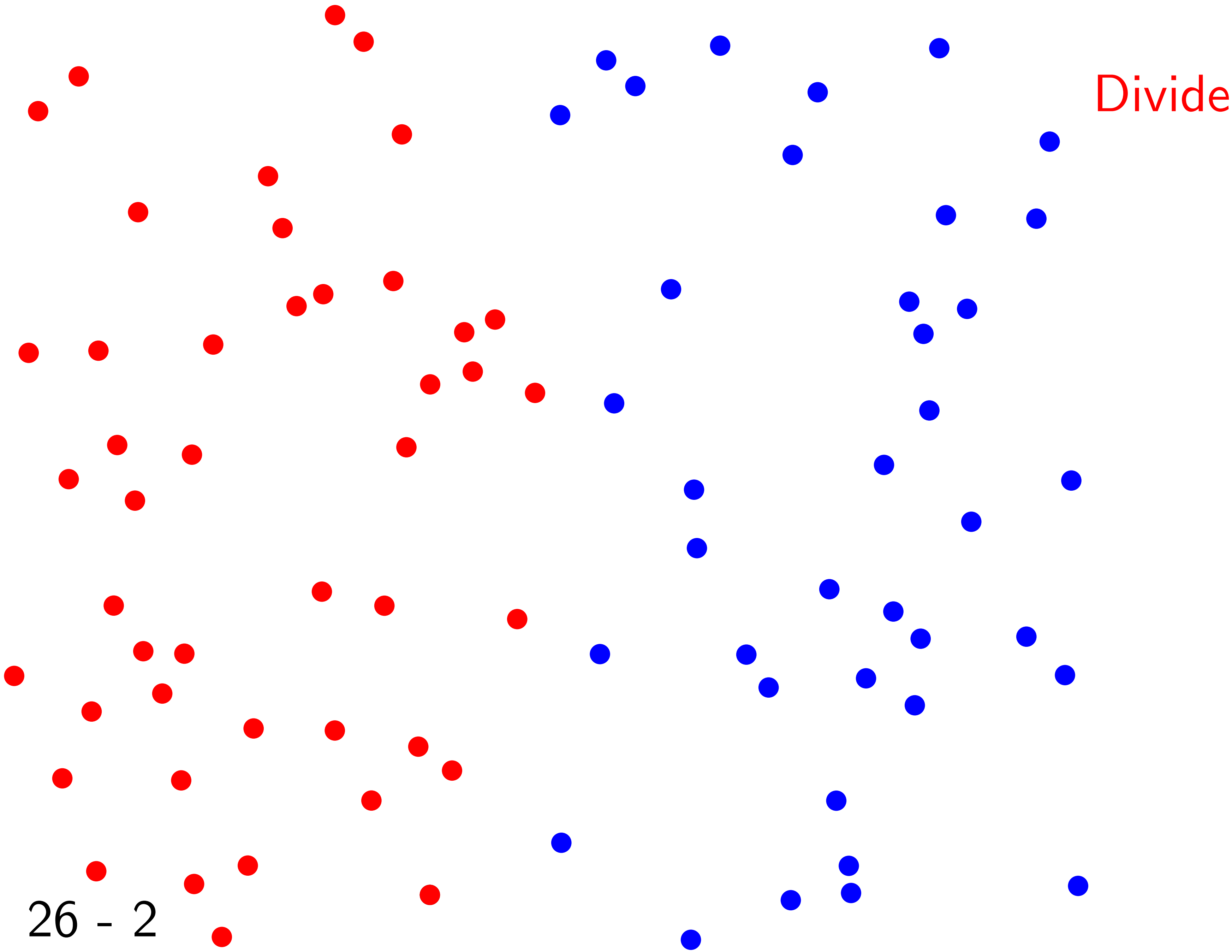
$$O(n \log n)$$

Algorithm: divide and conquer

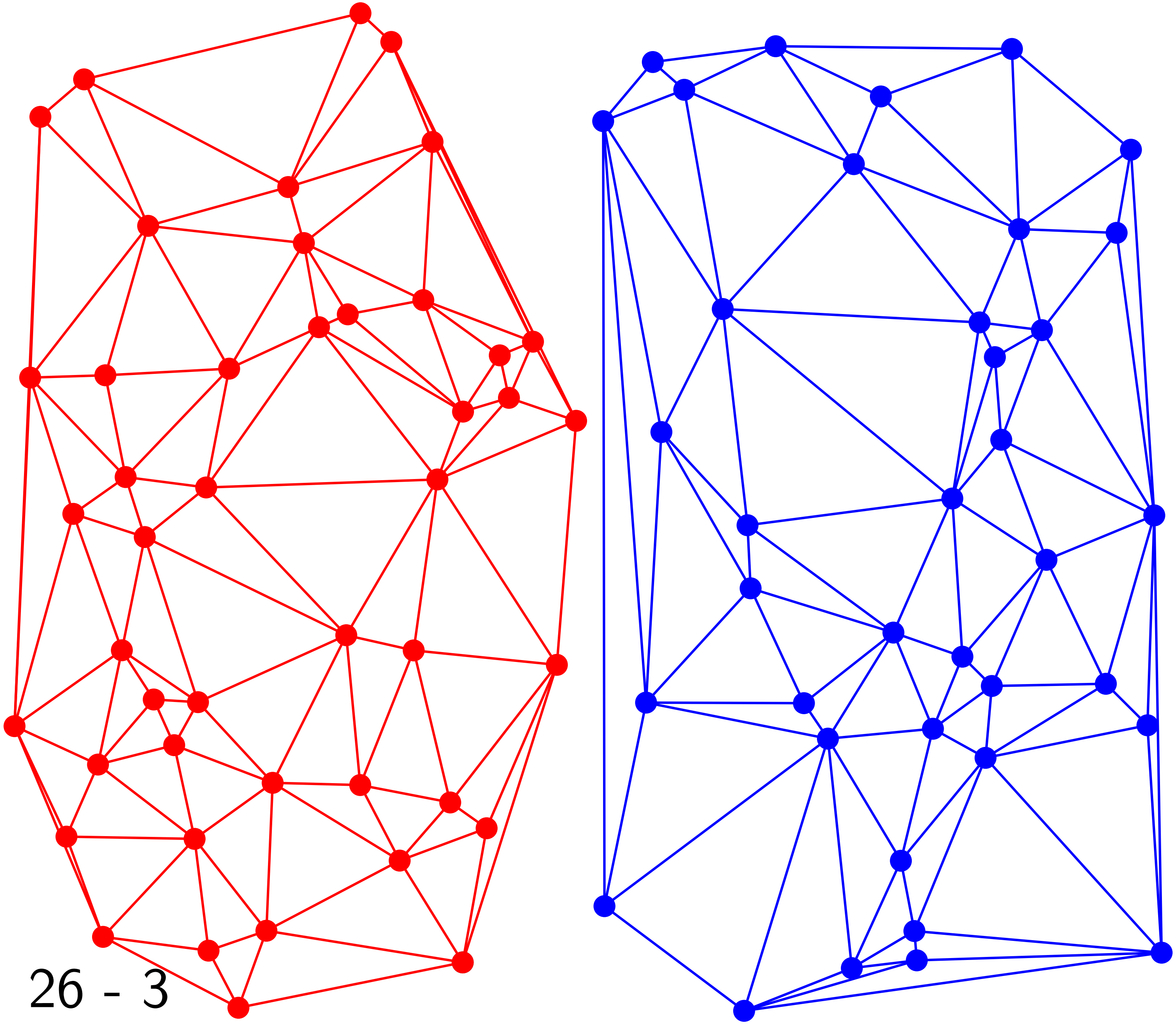
Delaunay Triangulation: divide & conquer (sketch)



Delaunay Triangulation: divide & conquer (sketch)



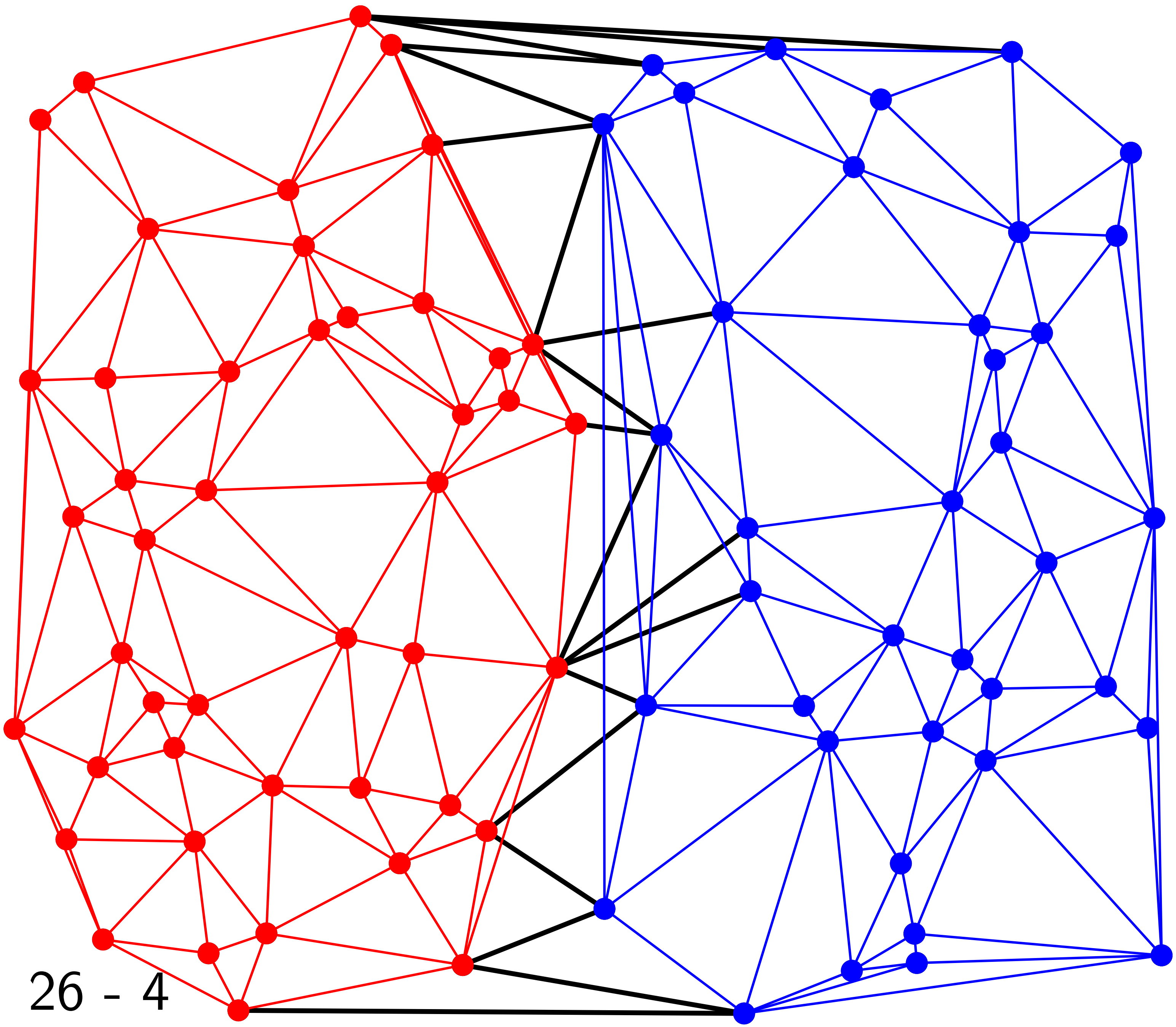
Delaunay Triangulation: divide & conquer (sketch)



Divide

Recurse

Delaunay Triangulation: divide & conquer (sketch)

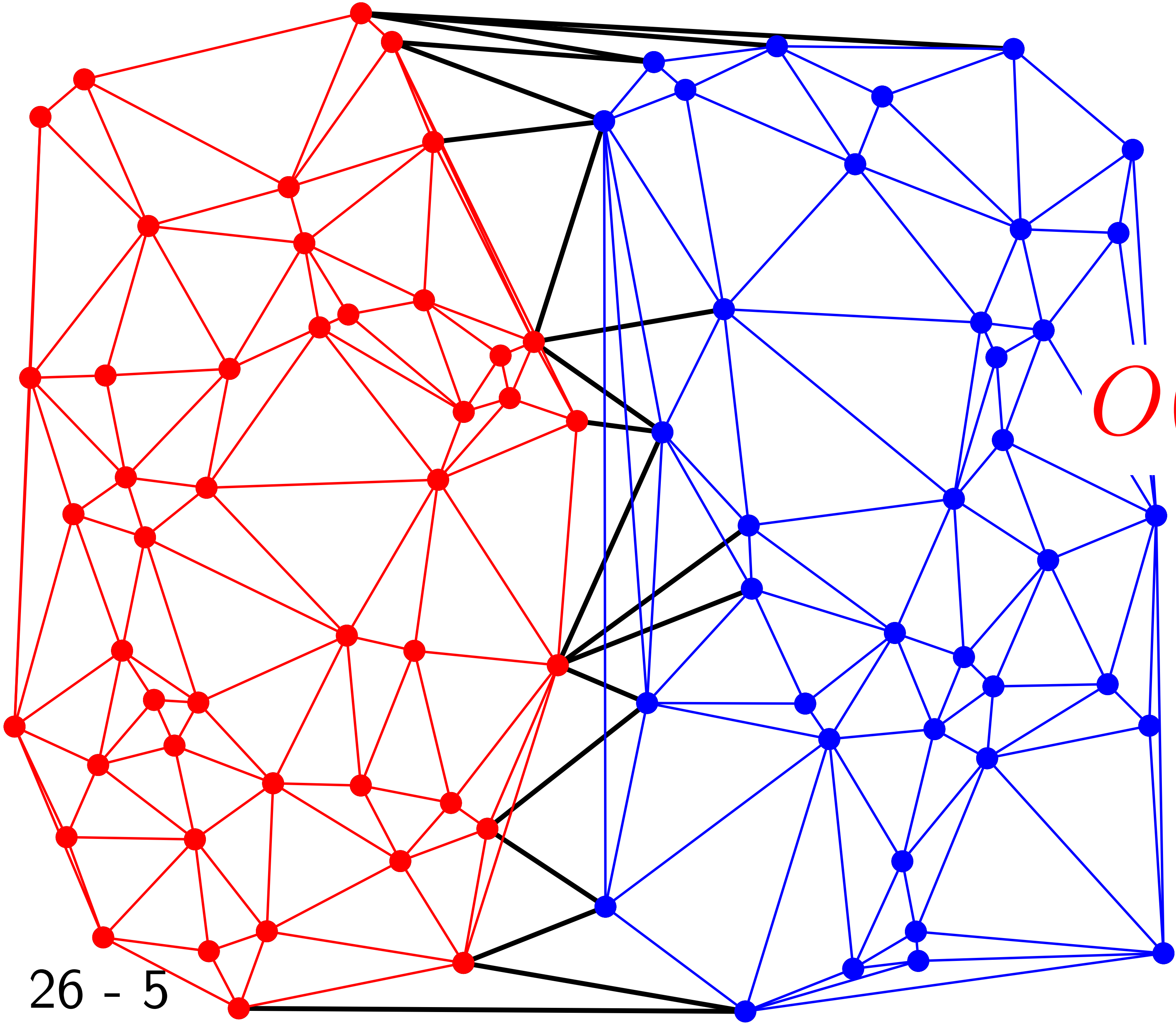


Divide

Recurse

Conquer

Delaunay Triangulation: divide & conquer (sketch)



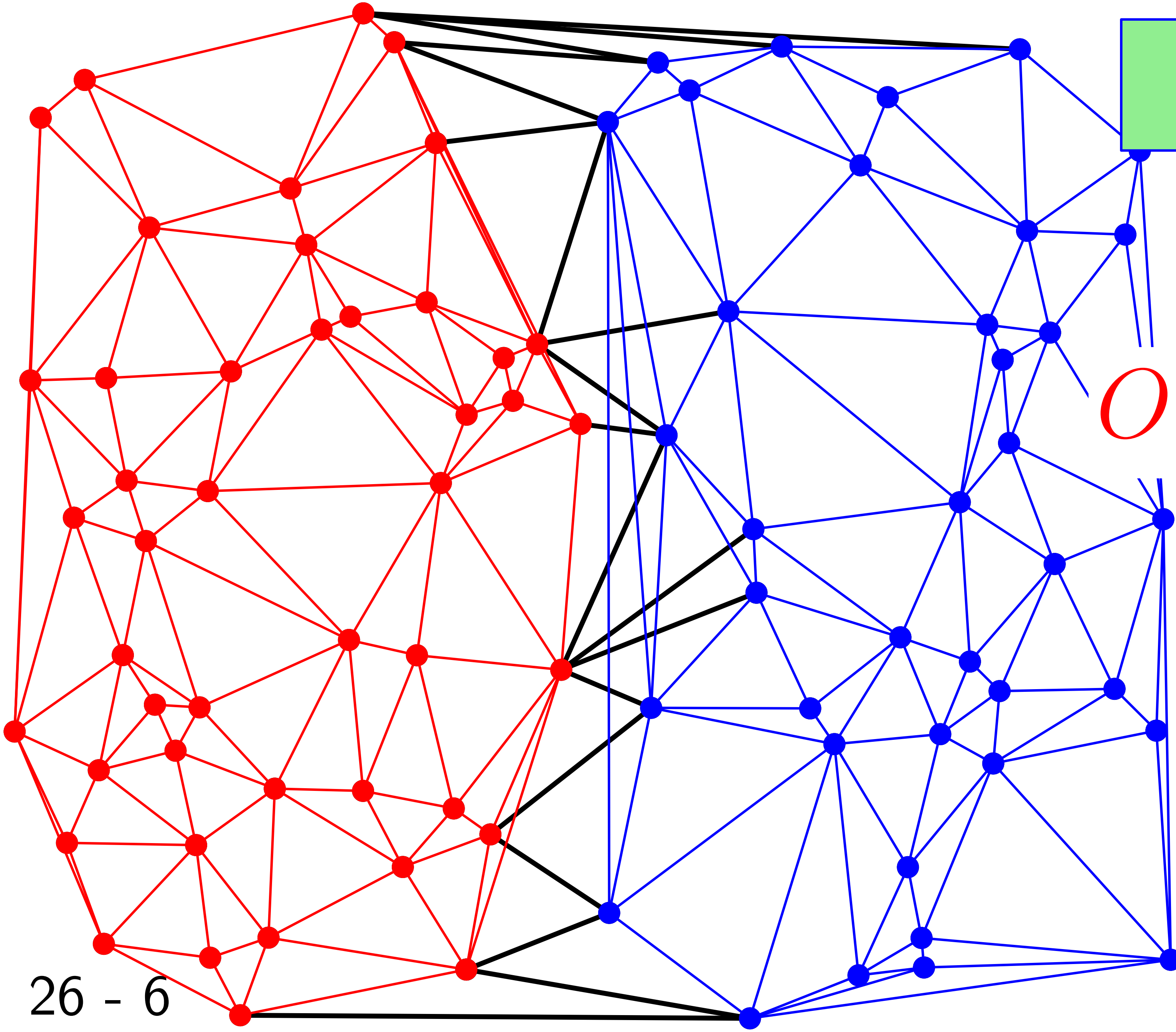
Divide

Recurse

Conquer

$$O(n \log n)$$

Delaunay Triangulation: divide & conquer (sketch)



Divide

Recurse

Conquer

$$O(n \log n)$$

balanced

linear time

easier conquer

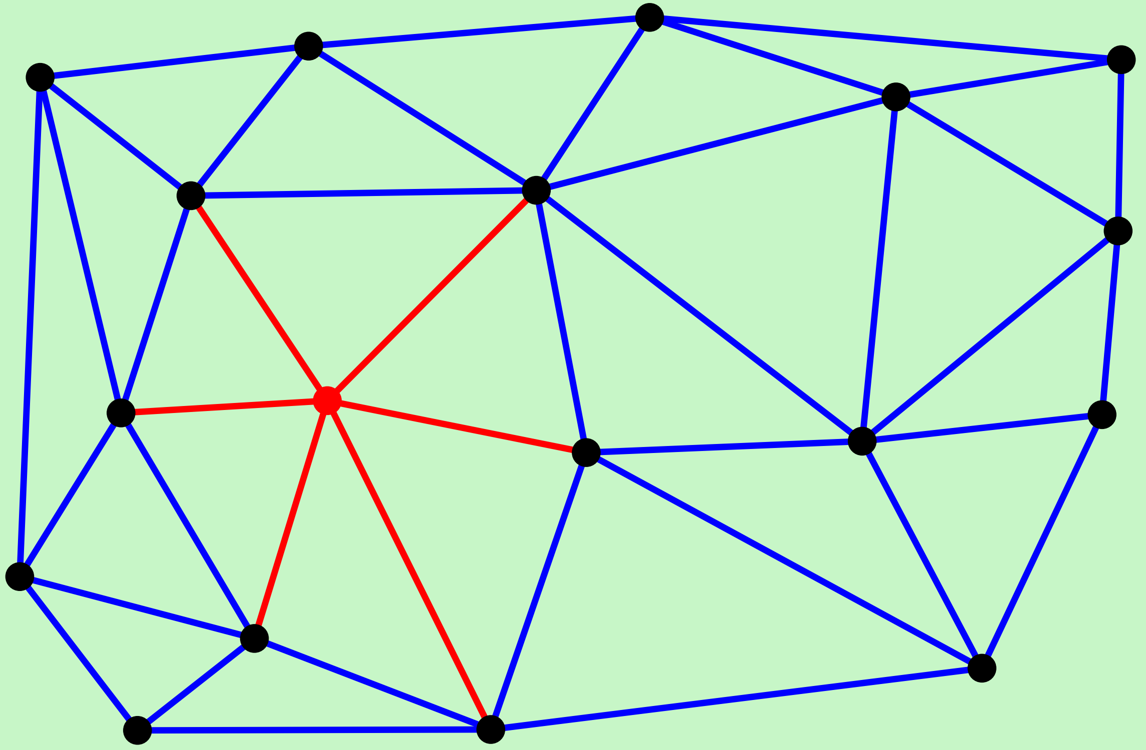
Deleting a point

Delaunay Triangulation: deletion algorithm (sketch)

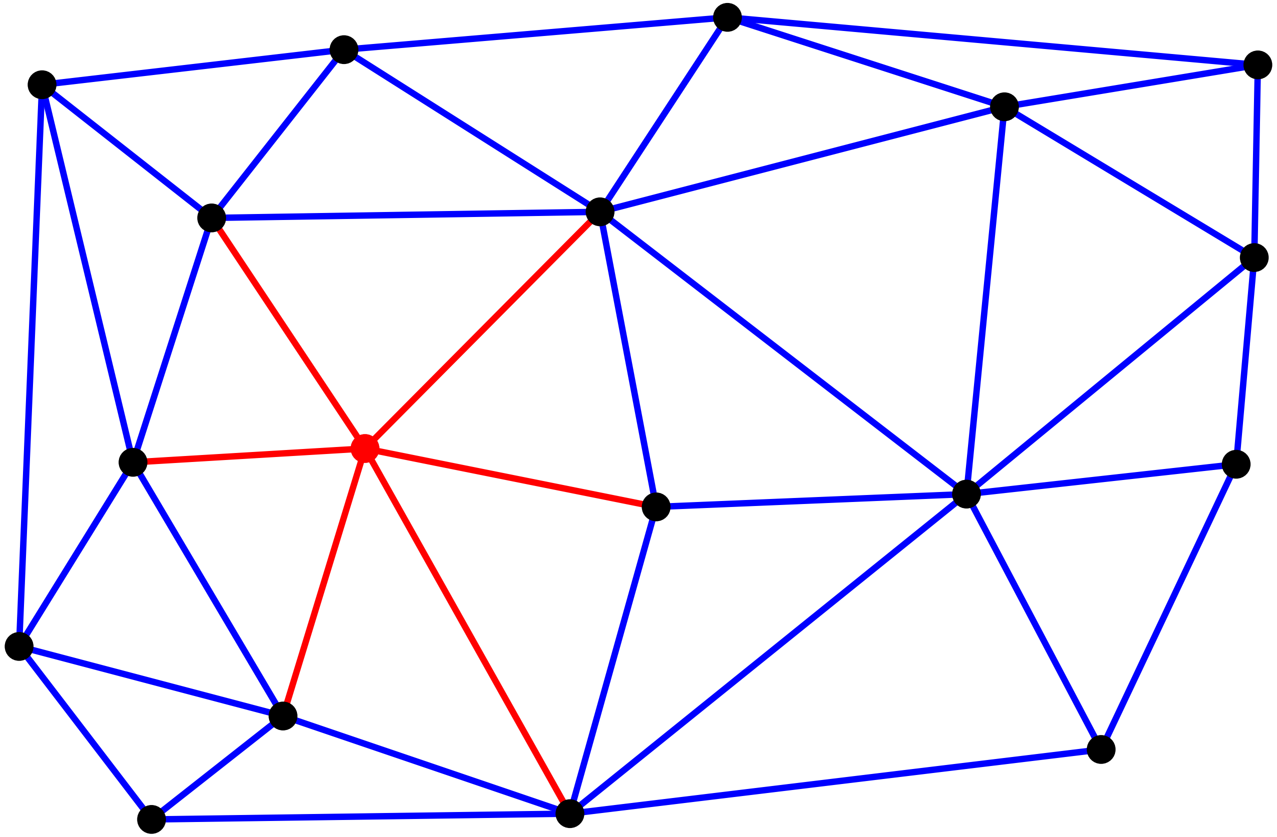
Delaunay Triangulation: deletion algorithm (sketch)

Delaunay Triangulation: incremental algorithm

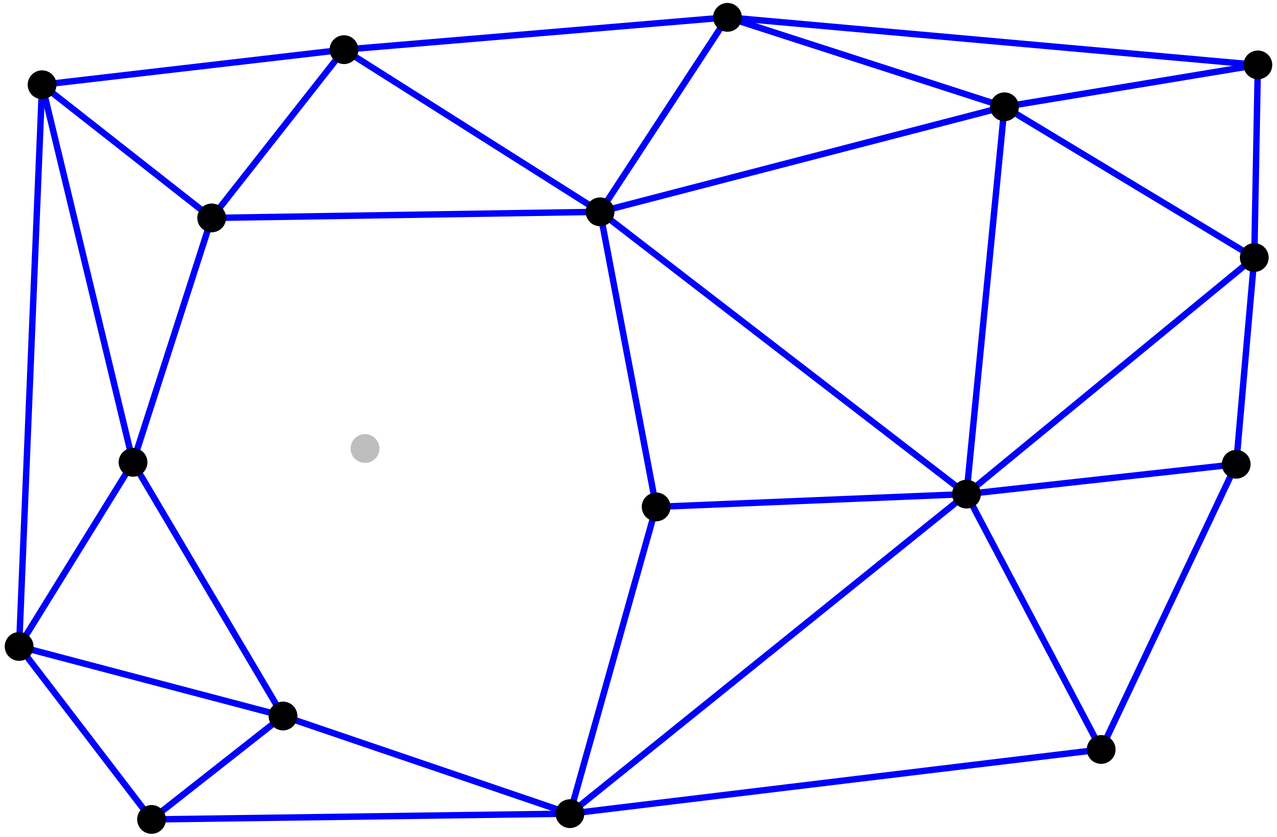
New point



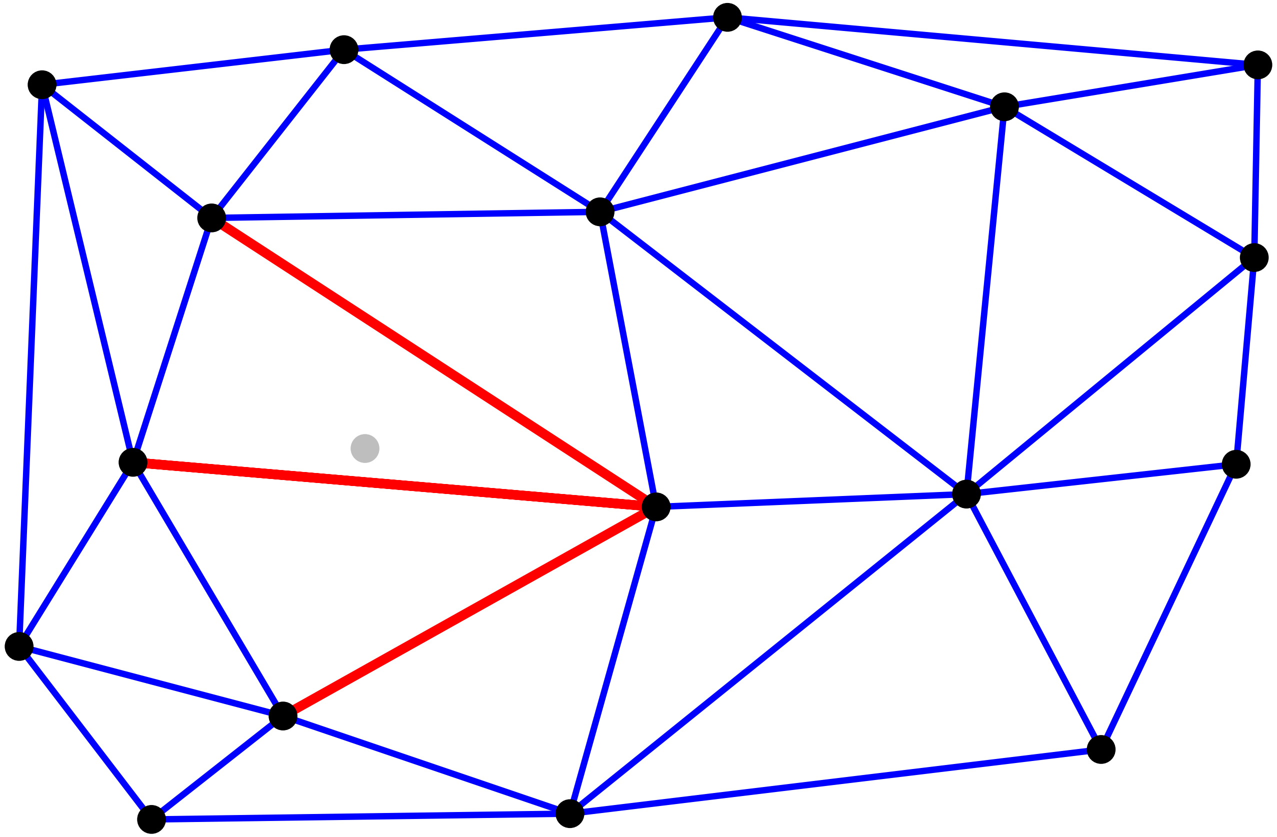
Delaunay Triangulation: deletion algorithm (sketch)



Delaunay Triangulation: deletion algorithm (sketch)

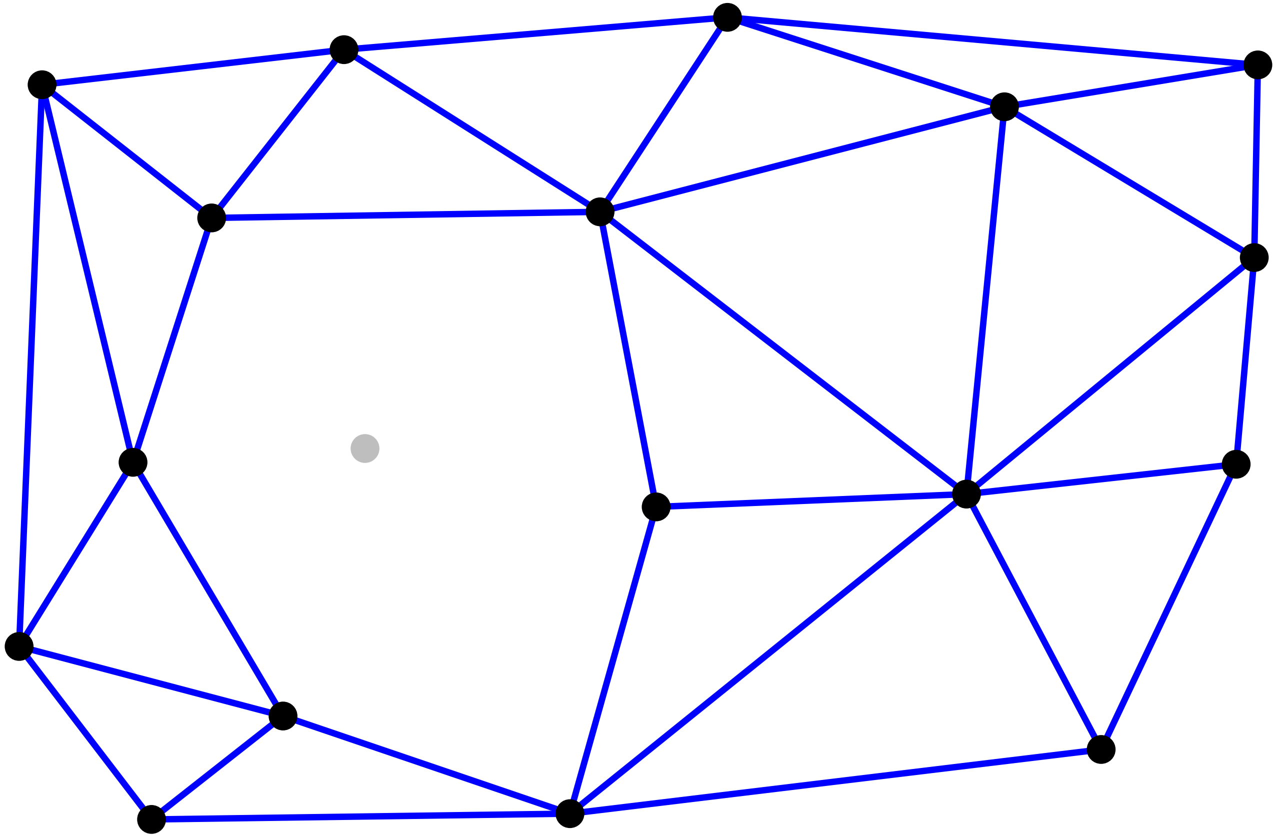


Delaunay Triangulation: deletion algorithm (sketch)



Delaunay Triangulation: deletion algorithm (sketch)

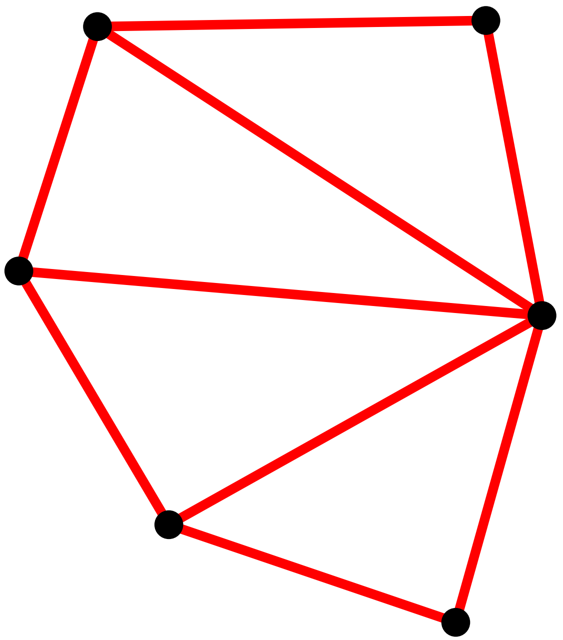
Extract hole

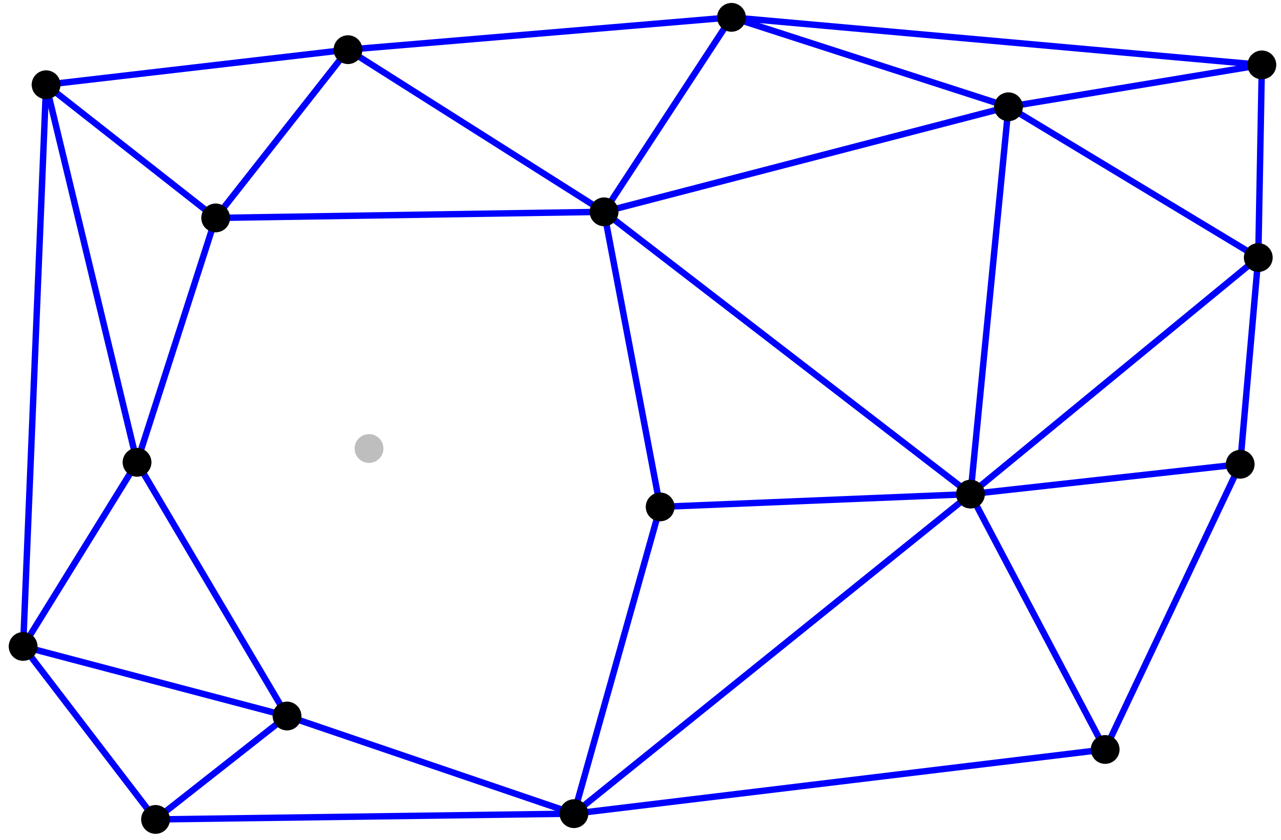


Delaunay Triangulation: deletion algorithm (sketch)

Extract hole

Triangulate

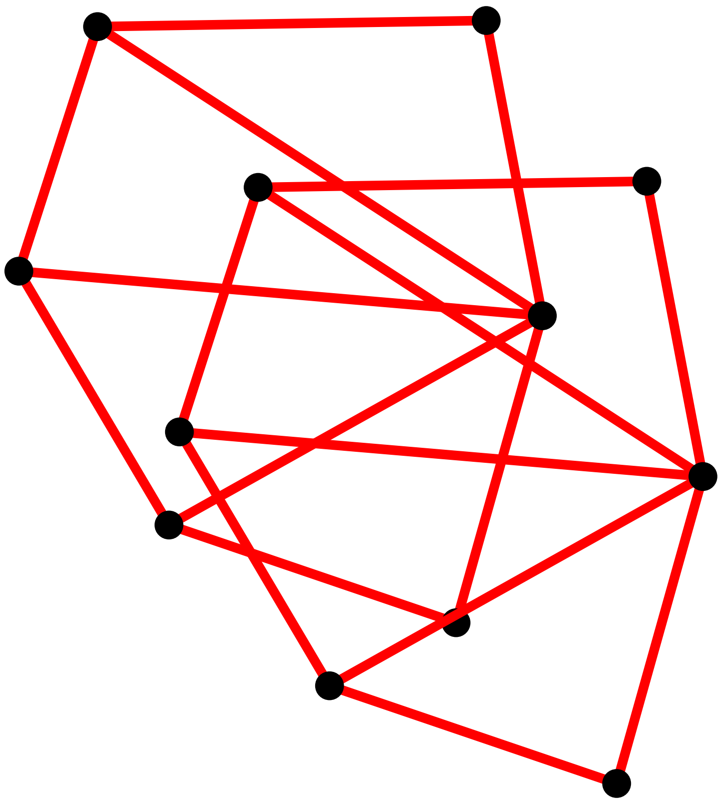


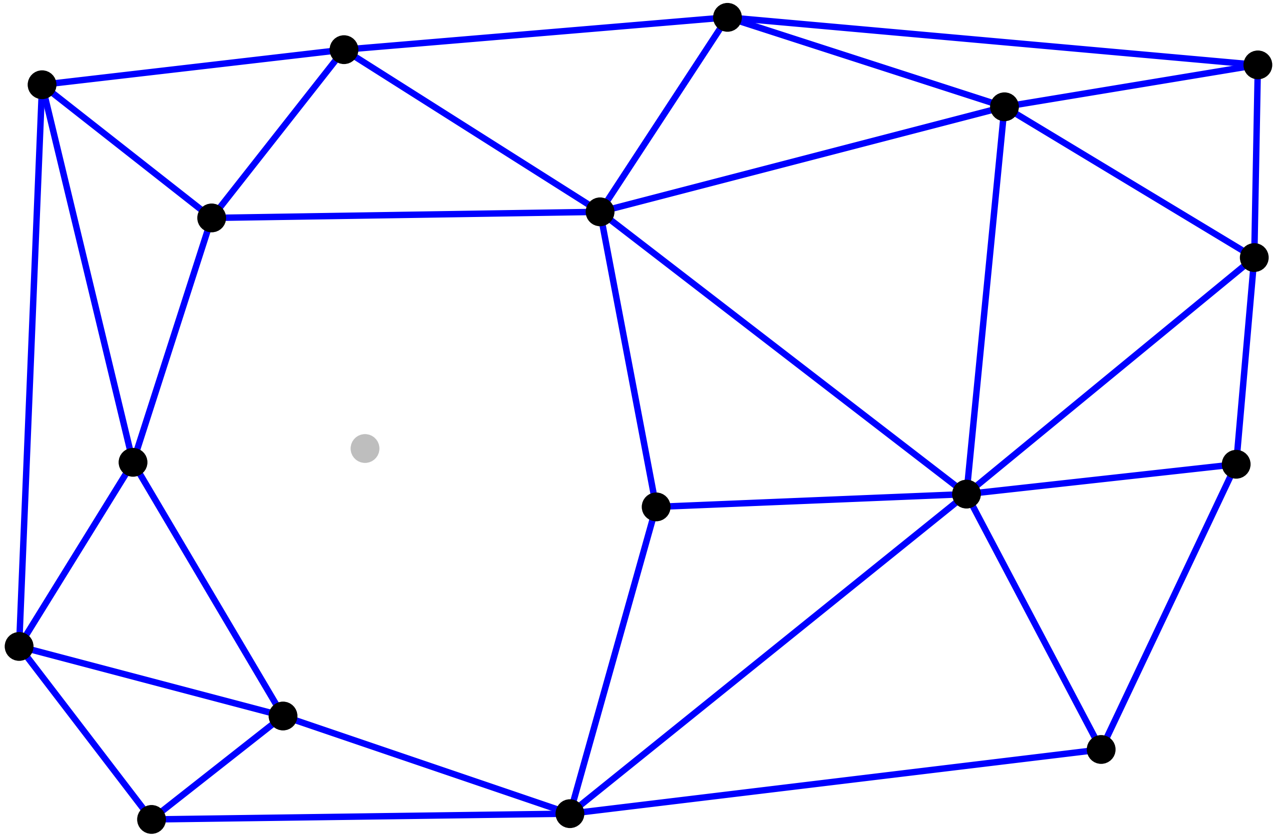


Delaunay Triangulation: deletion algorithm (sketch)

Extract hole

Triangulate

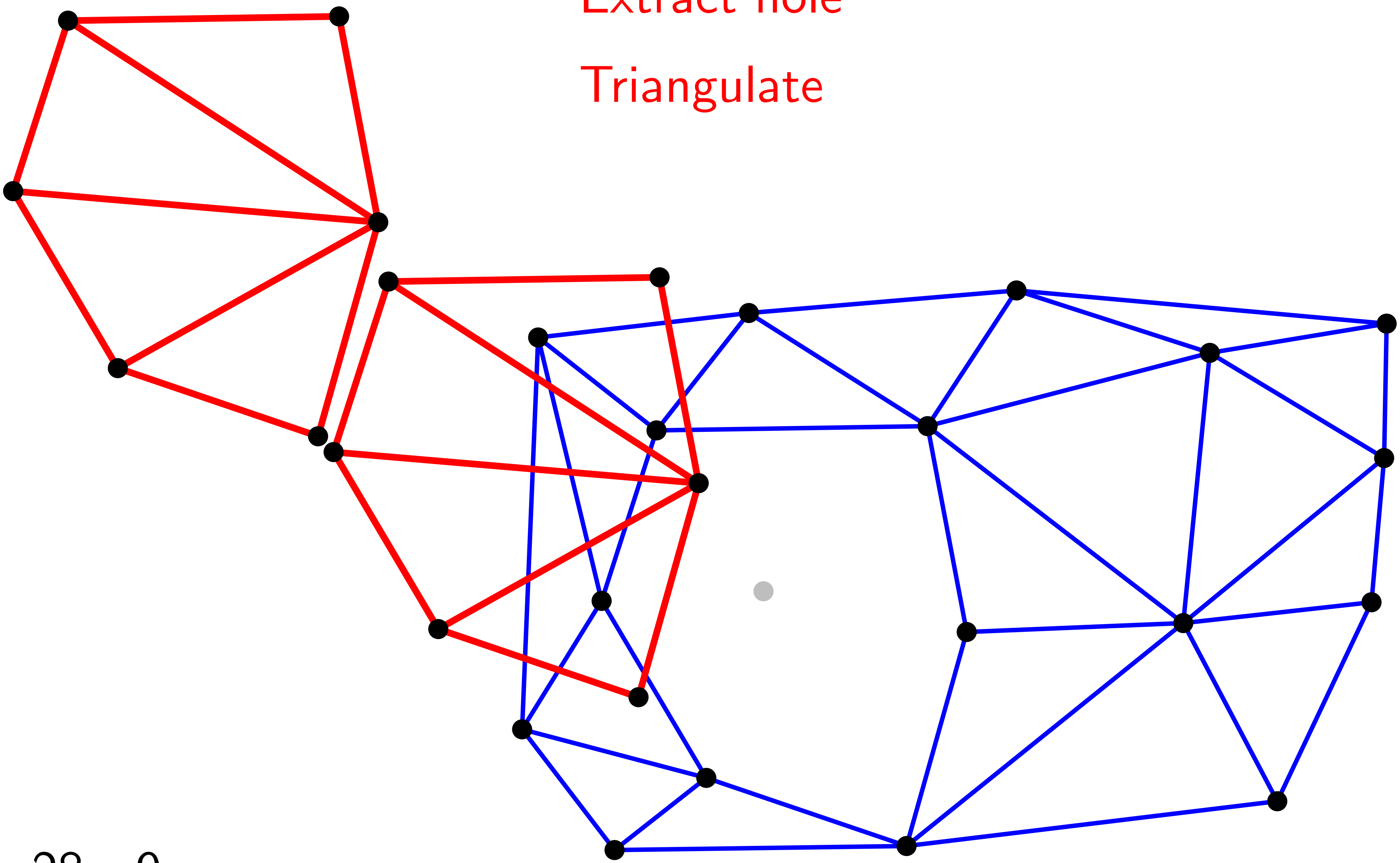




Delaunay Triangulation: deletion algorithm (sketch)

Extract hole

Triangulate

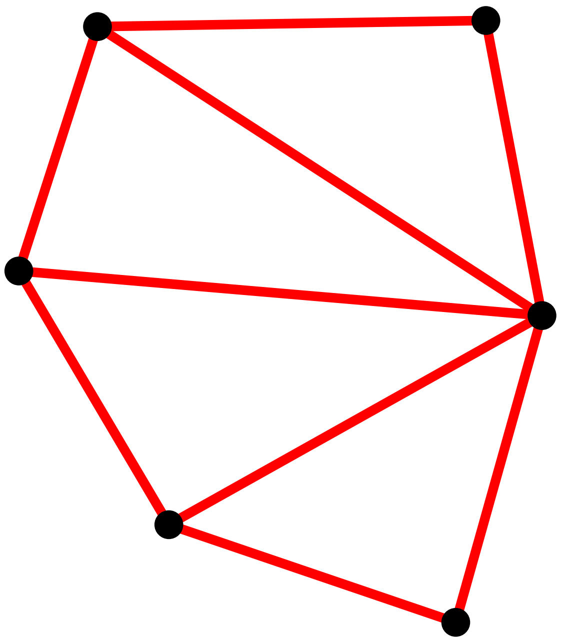


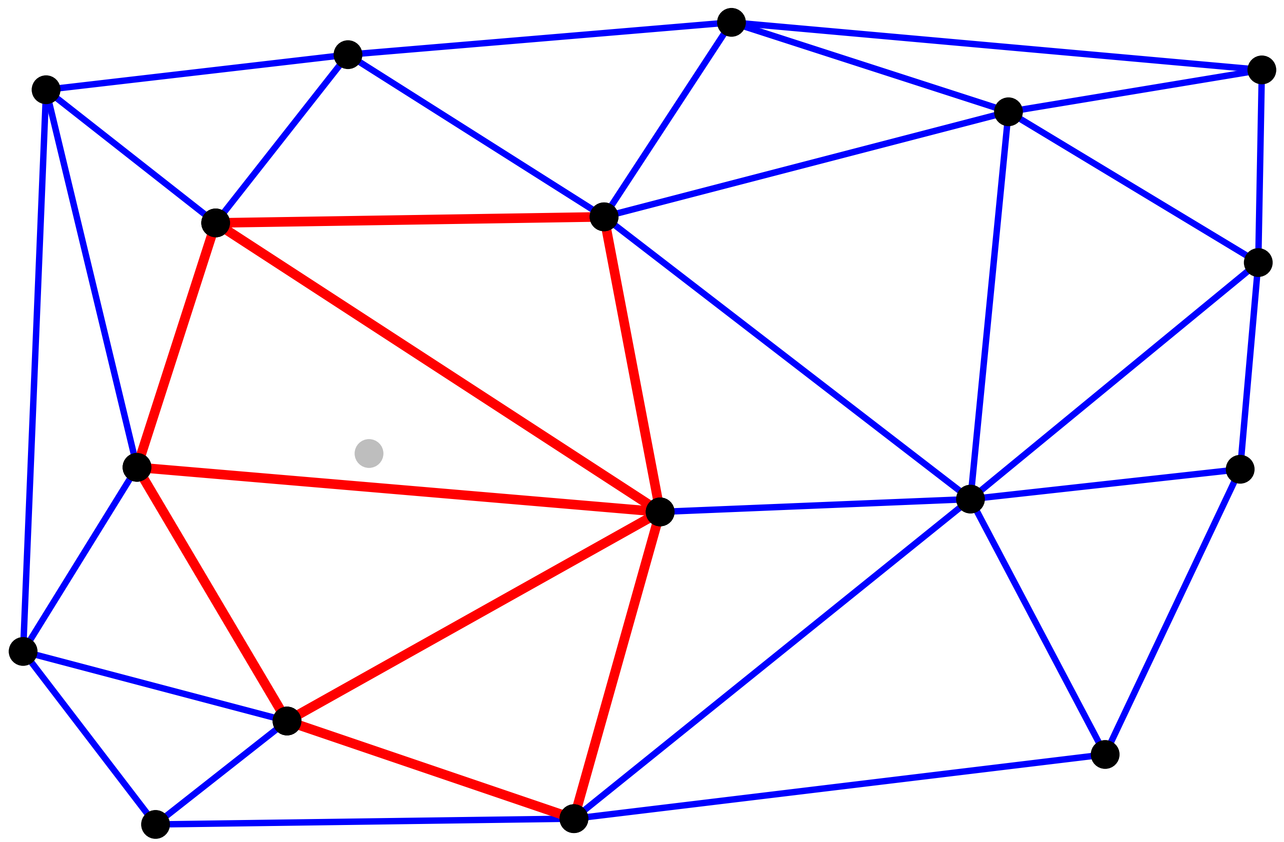
Delaunay Triangulation: deletion algorithm (sketch)

Extract hole

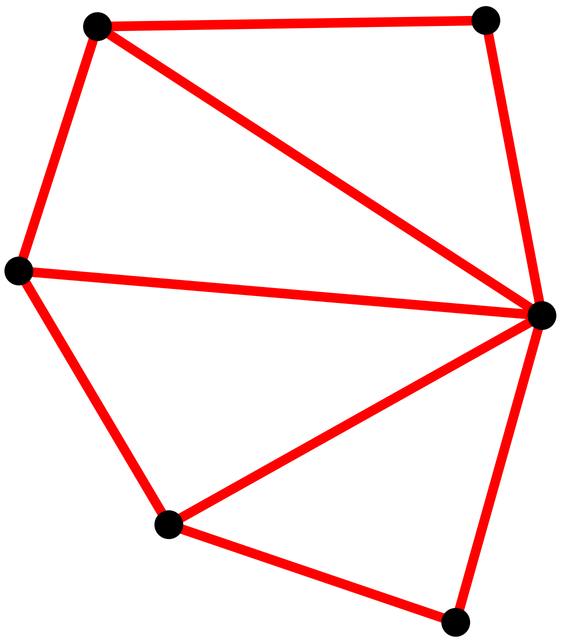
Triangulate

and sew





Delaunay Triangulation: deletion algorithm (sketch)



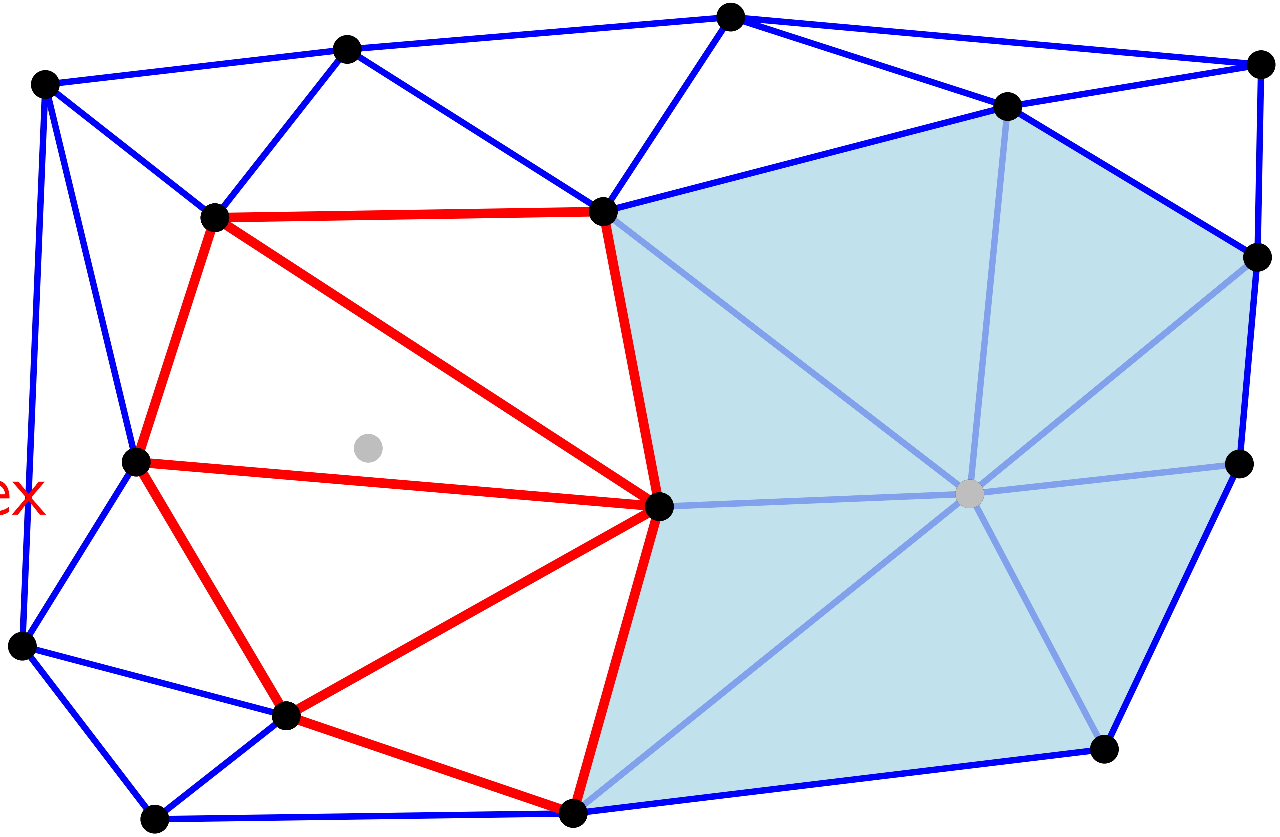
Extract hole

Triangulate

and sew

Be careful

Hole may be not convex

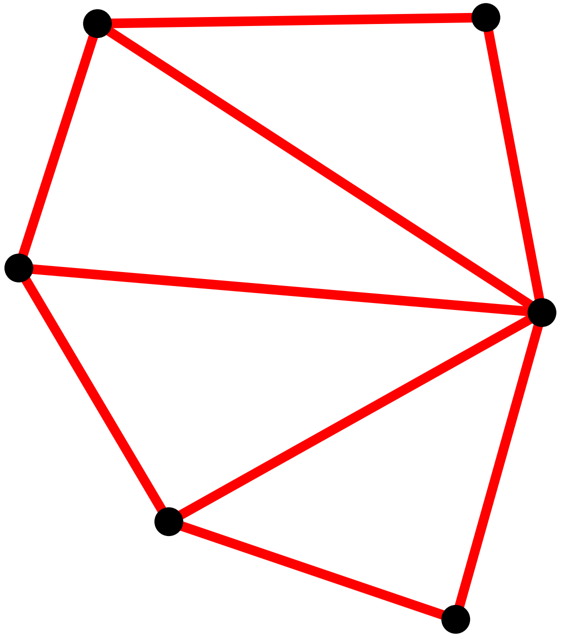


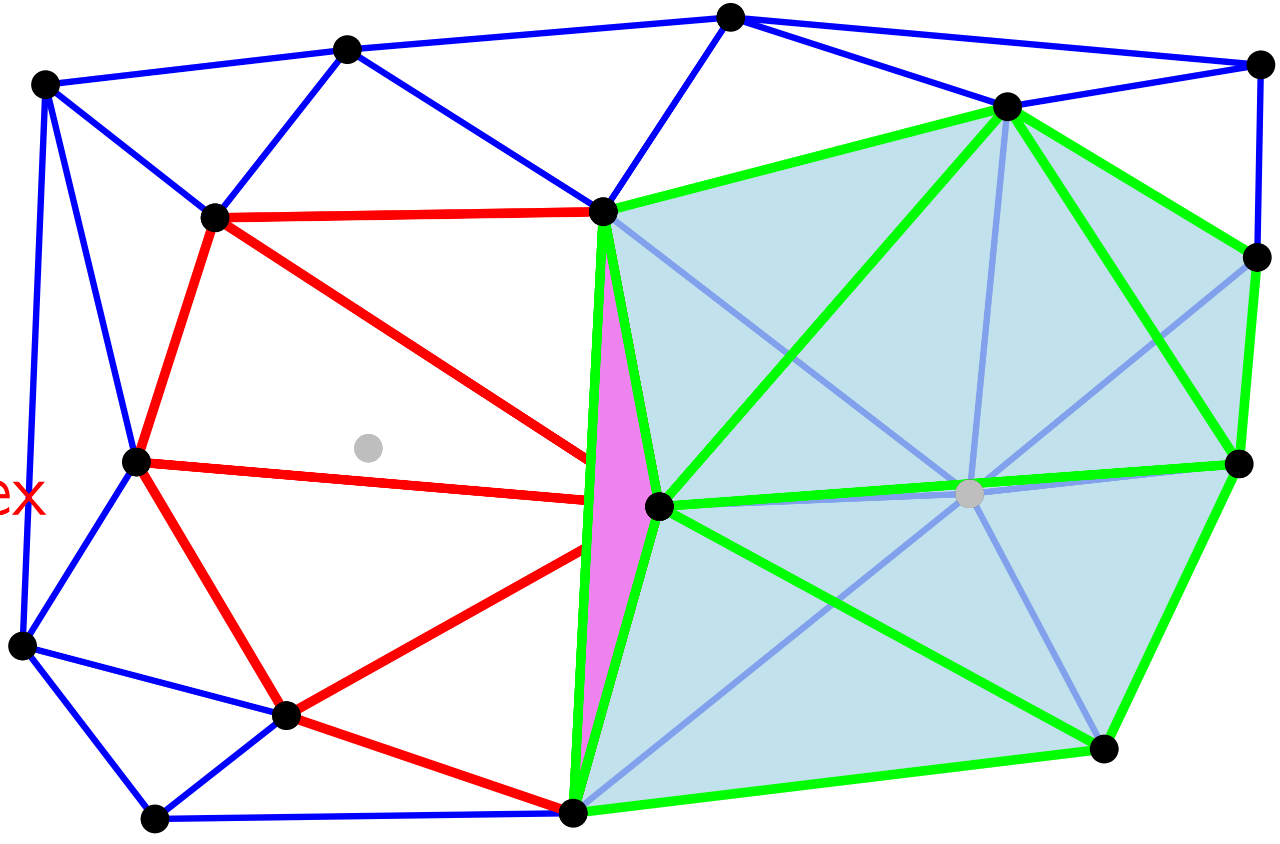
Delaunay Triangulation: deletion algorithm (sketch)

Extract hole

Triangulate

and sew



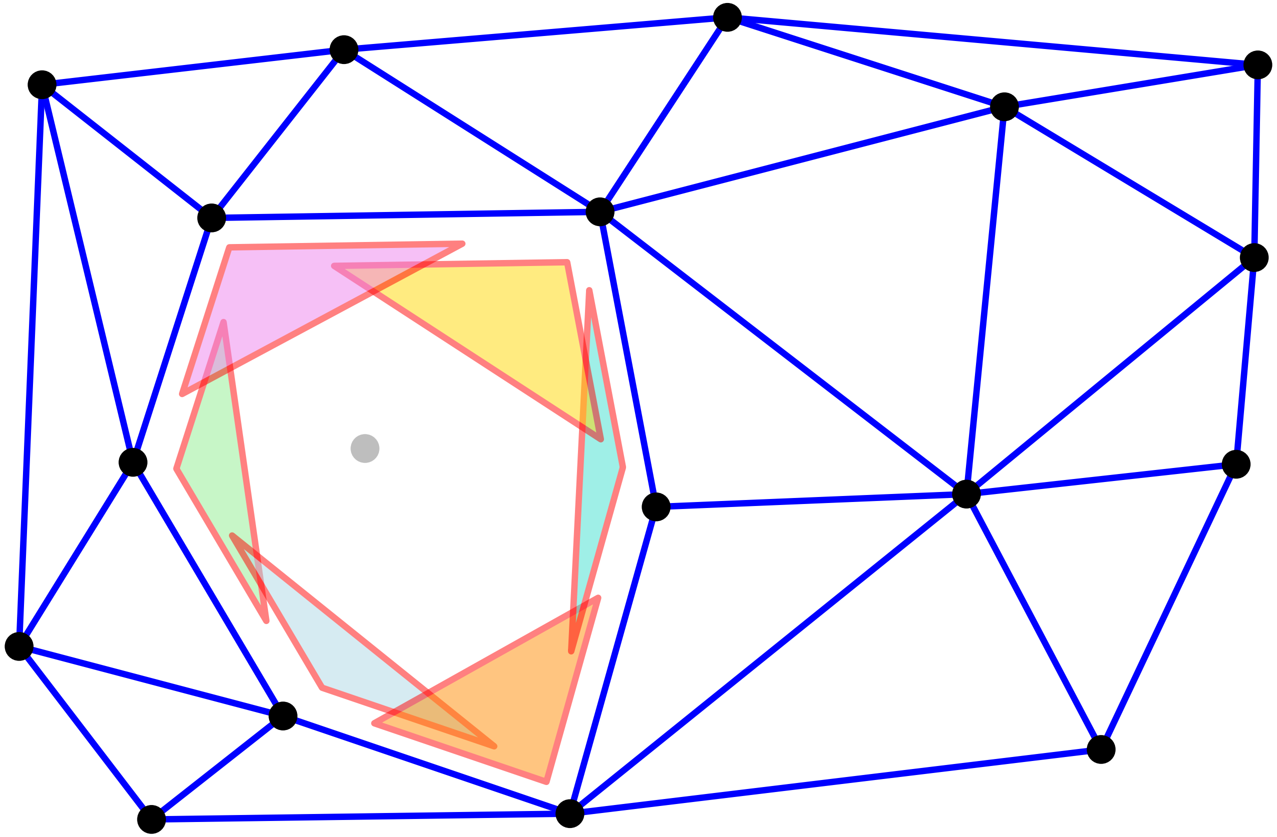


Be careful

Hole may be not convex

Delaunay Triangulation: deletion algorithm (sketch)

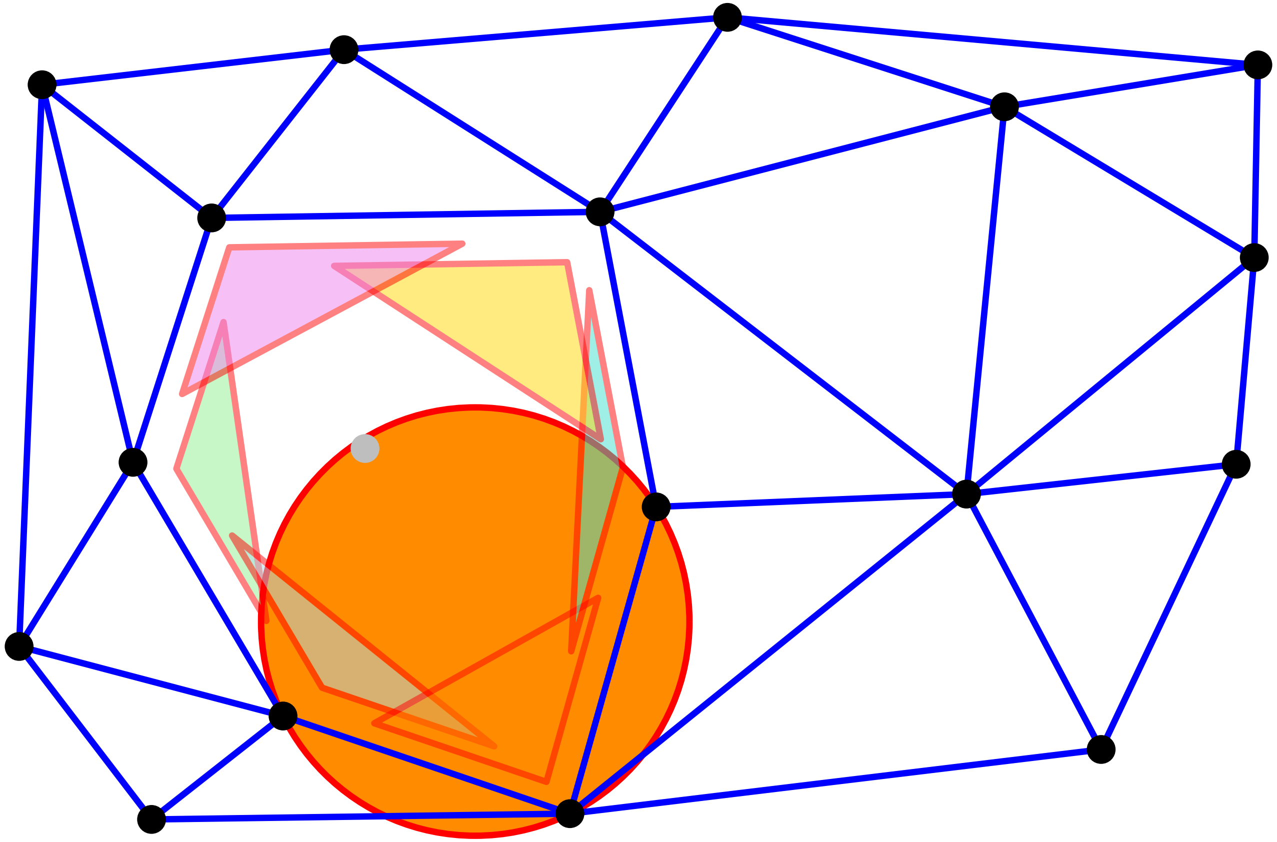
Ear queue



Delaunay Triangulation: deletion algorithm (sketch)

Ear queue

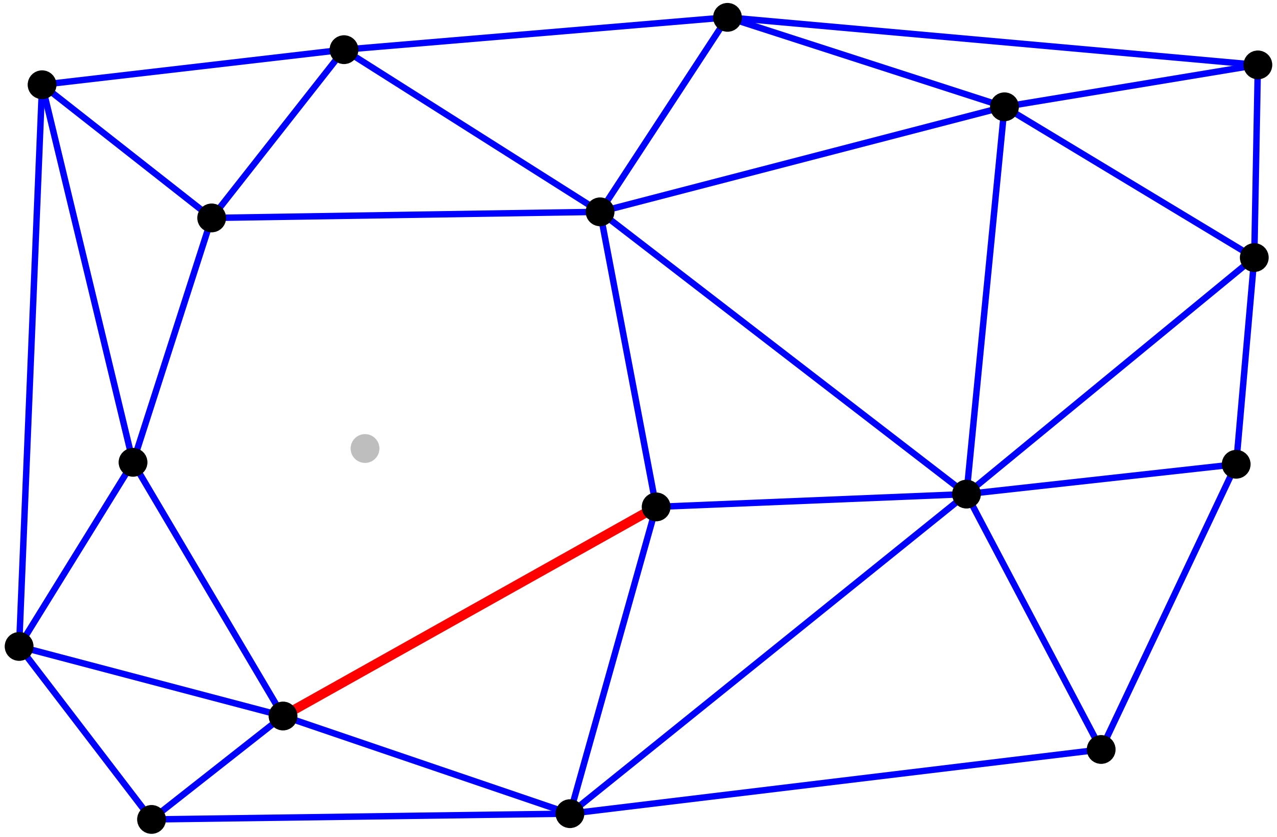
Ear with largest power is added



Delaunay Triangulation: deletion algorithm (sketch)

Ear queue

Ear with largest power is added

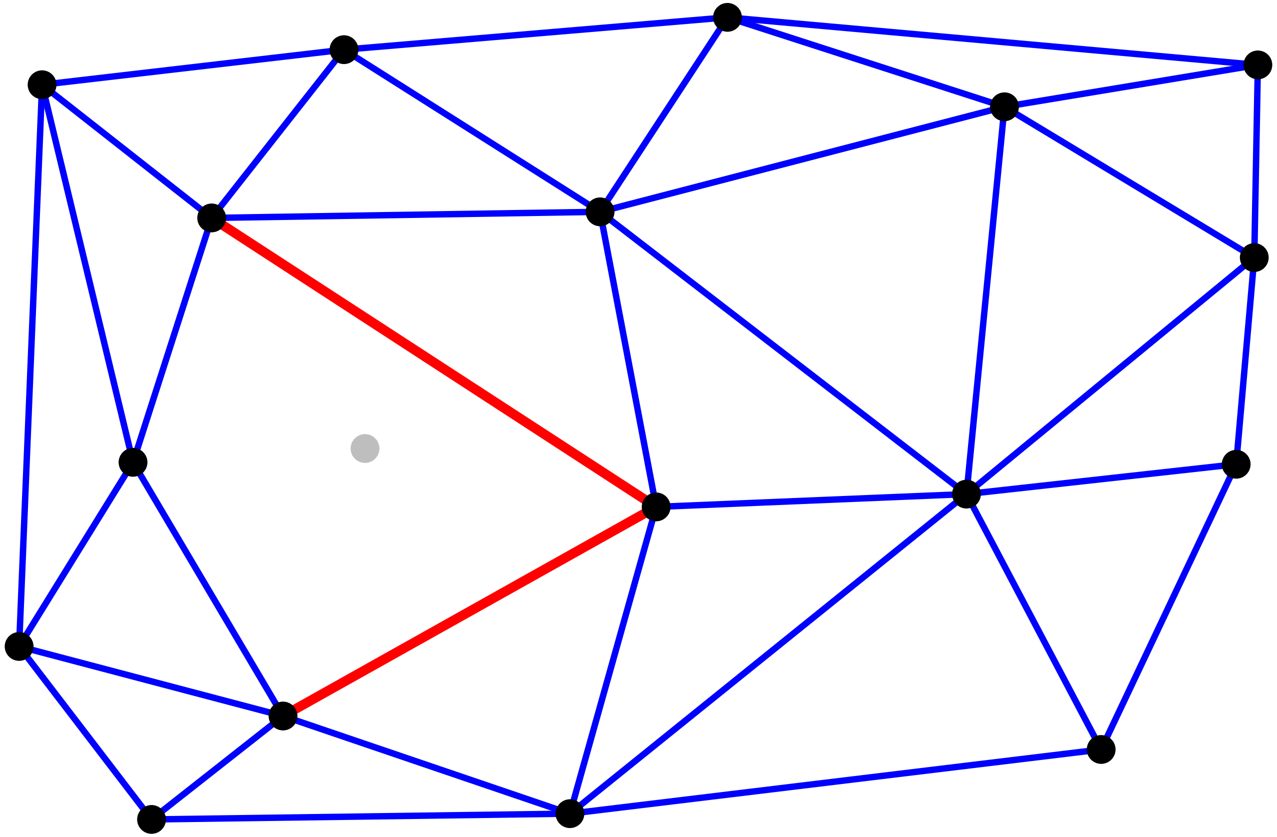


Delaunay Triangulation: deletion algorithm (sketch)

Ear queue

Ear with largest power is added

Iterate

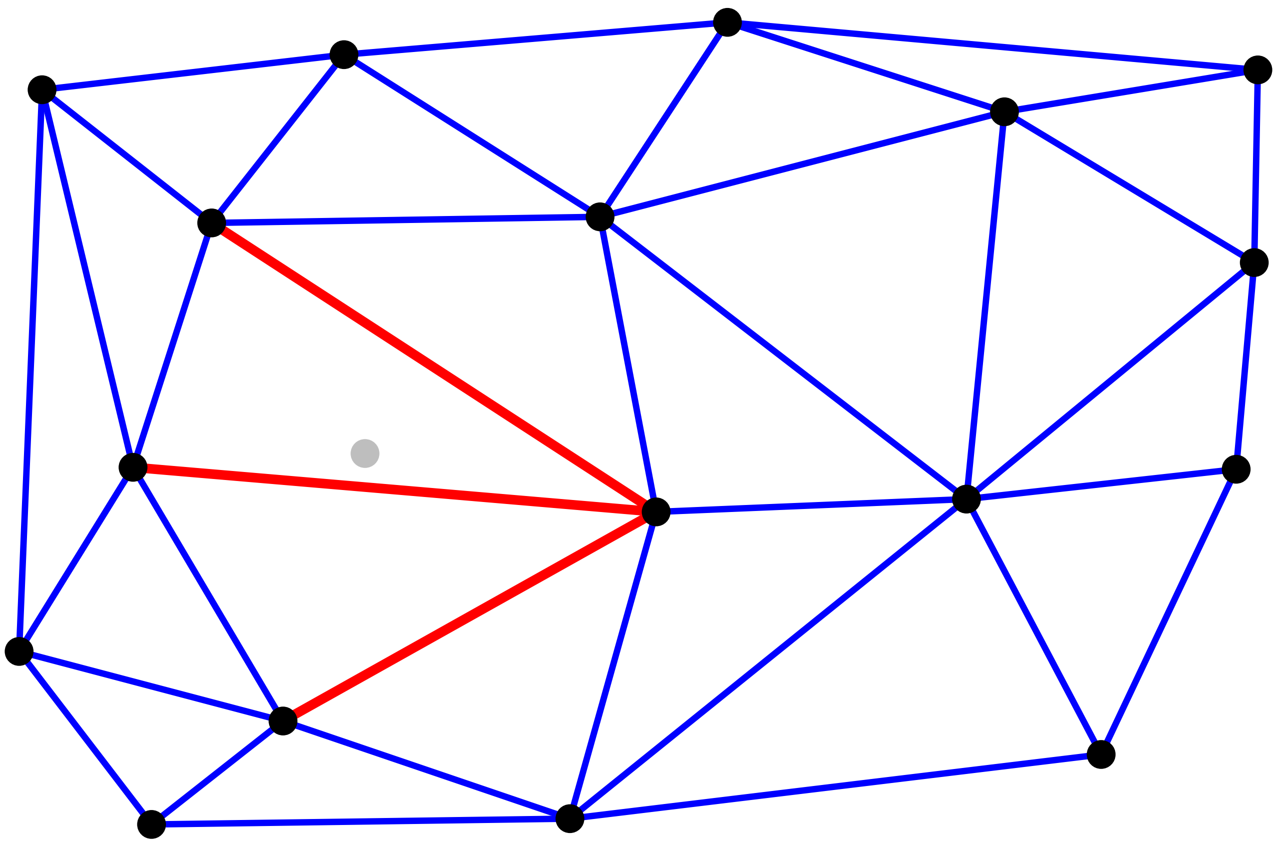


Delaunay Triangulation: deletion algorithm (sketch)

Ear queue

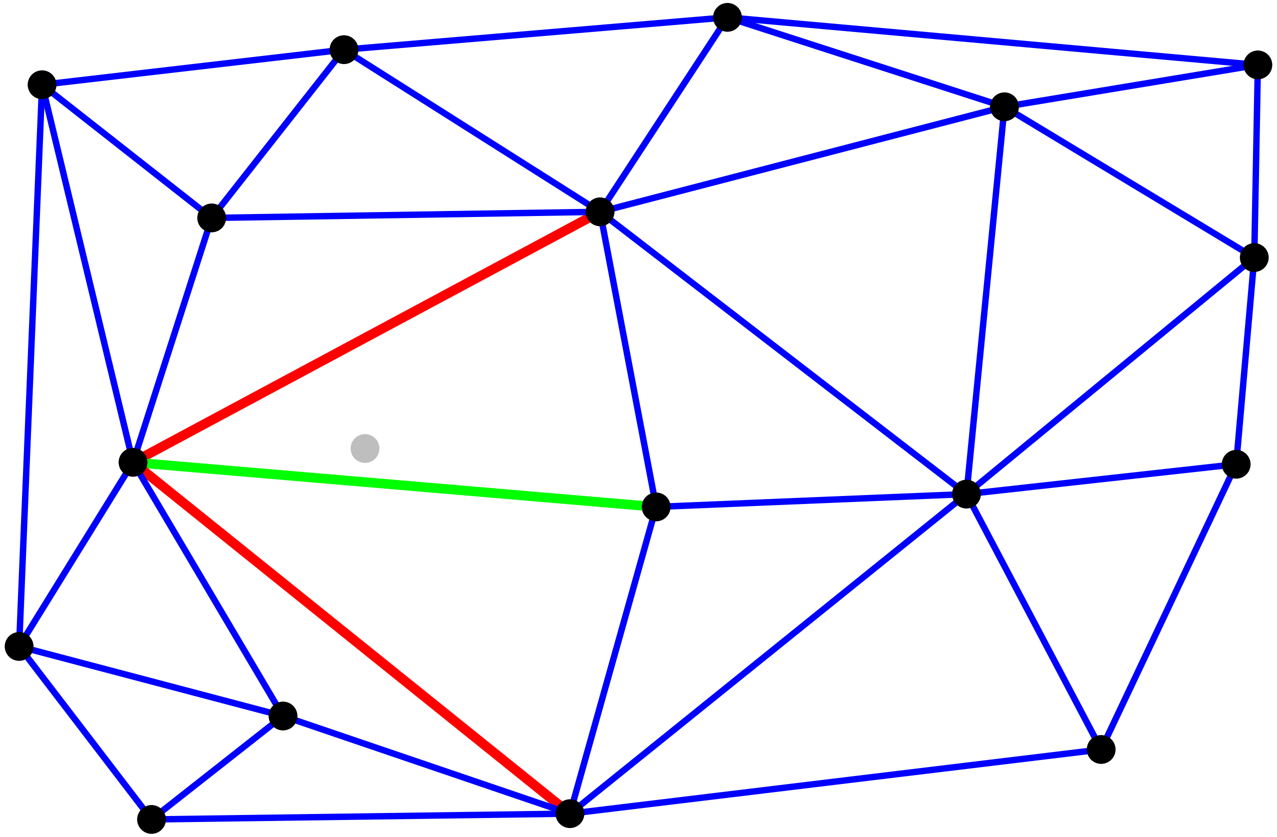
Ear with largest power is added

Iterate



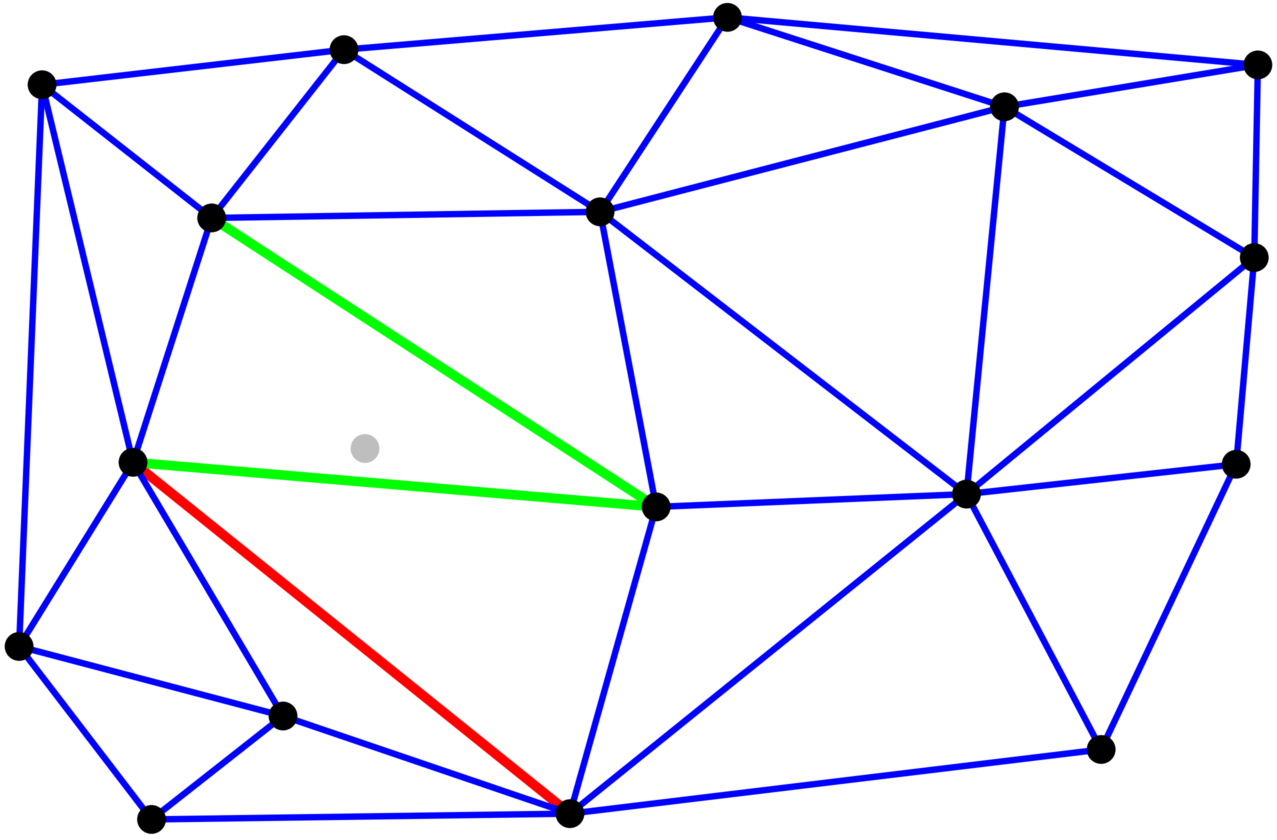
Delaunay Triangulation: deletion algorithm (sketch)

Triangulate and flip



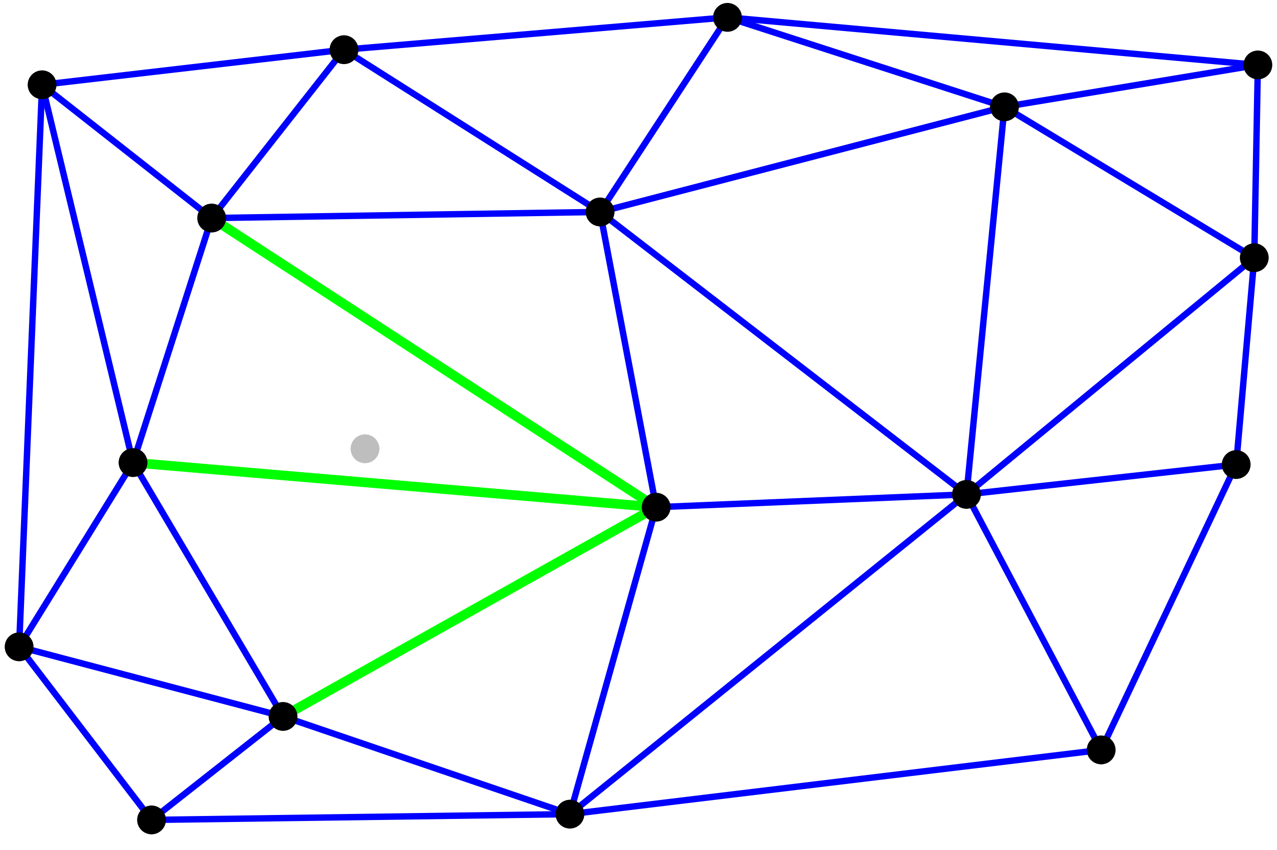
Delaunay Triangulation: deletion algorithm (sketch)

Triangulate and flip



Delaunay Triangulation: deletion algorithm (sketch)

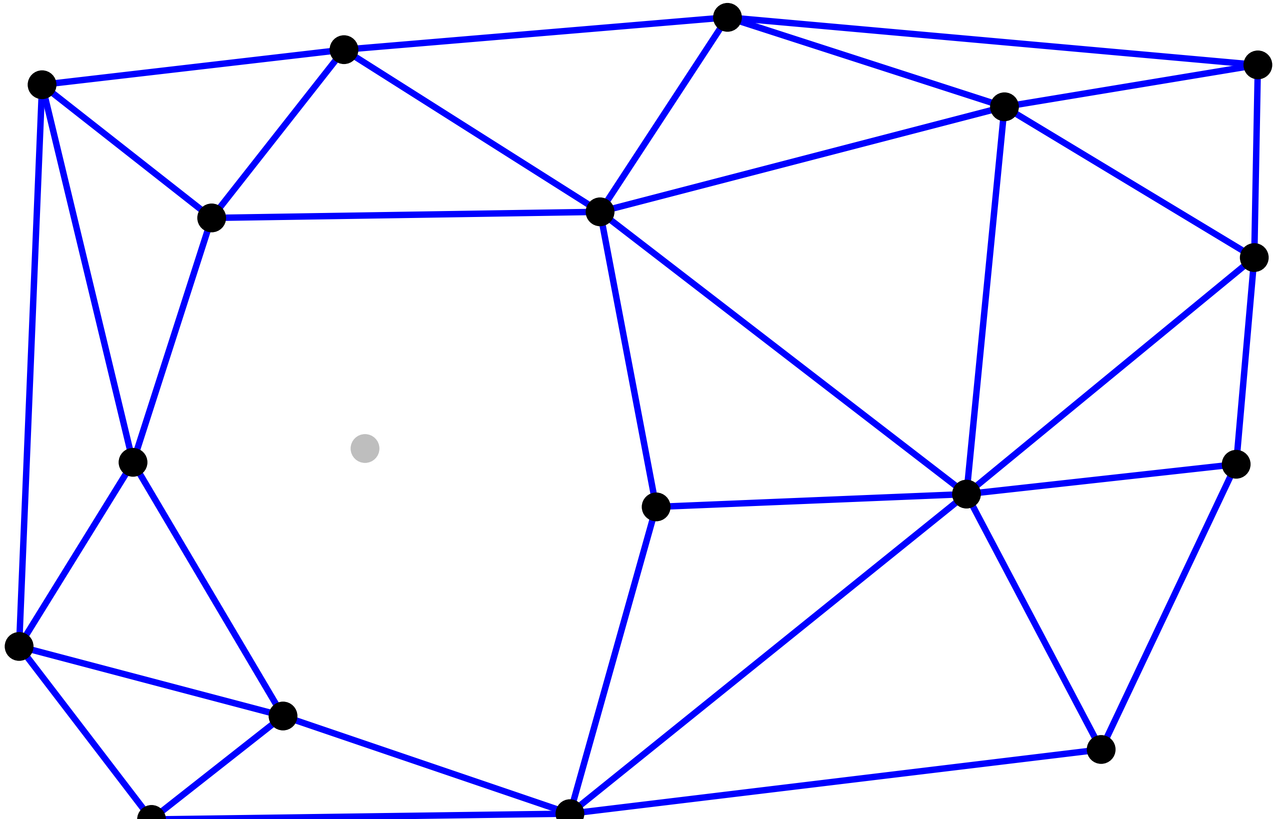
Triangulate and flip



for degree ≥ 8

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes



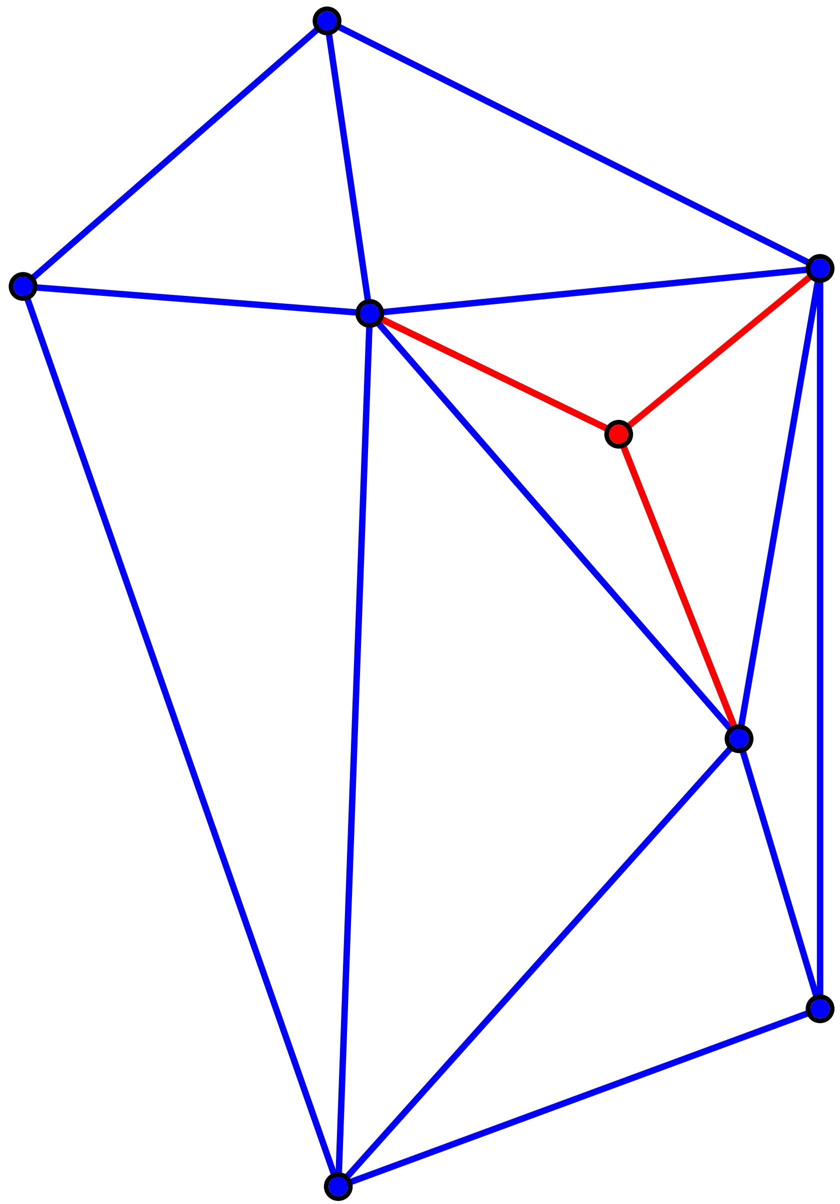
for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 3

nothing to do

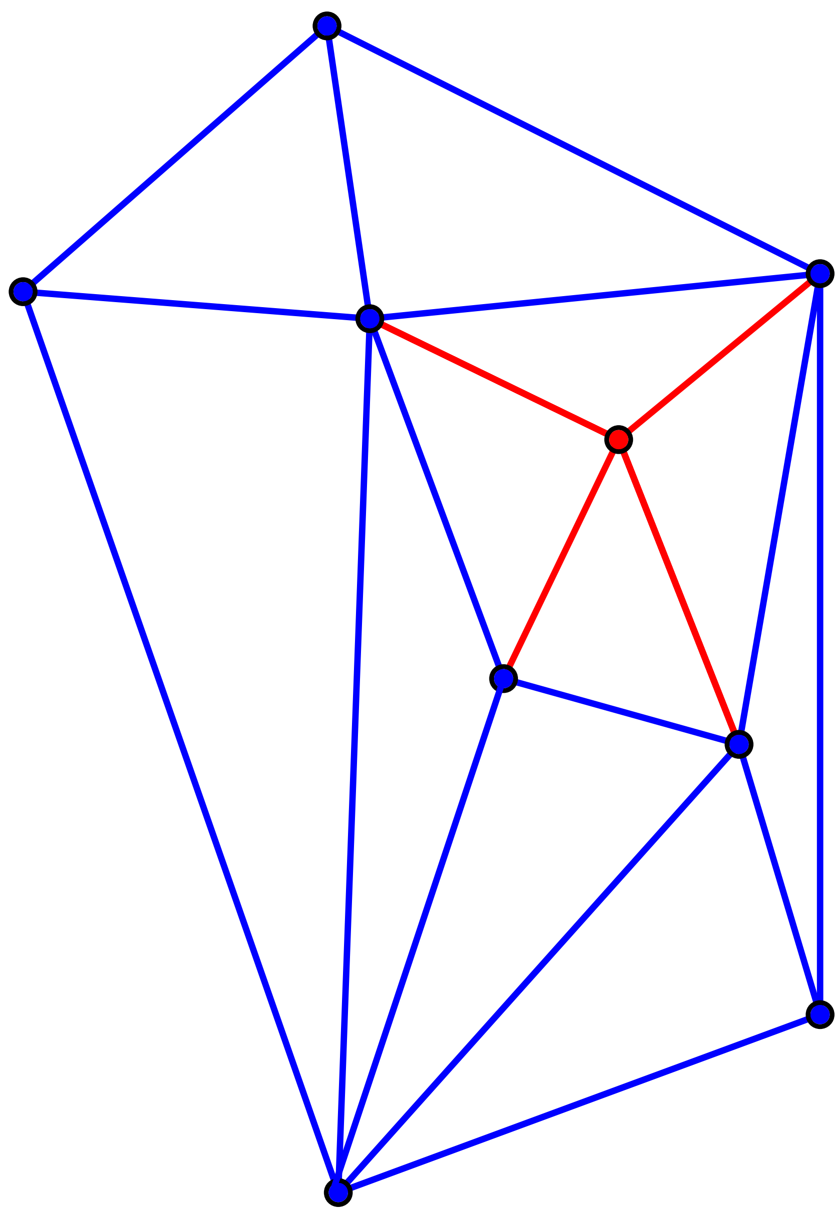


for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 4



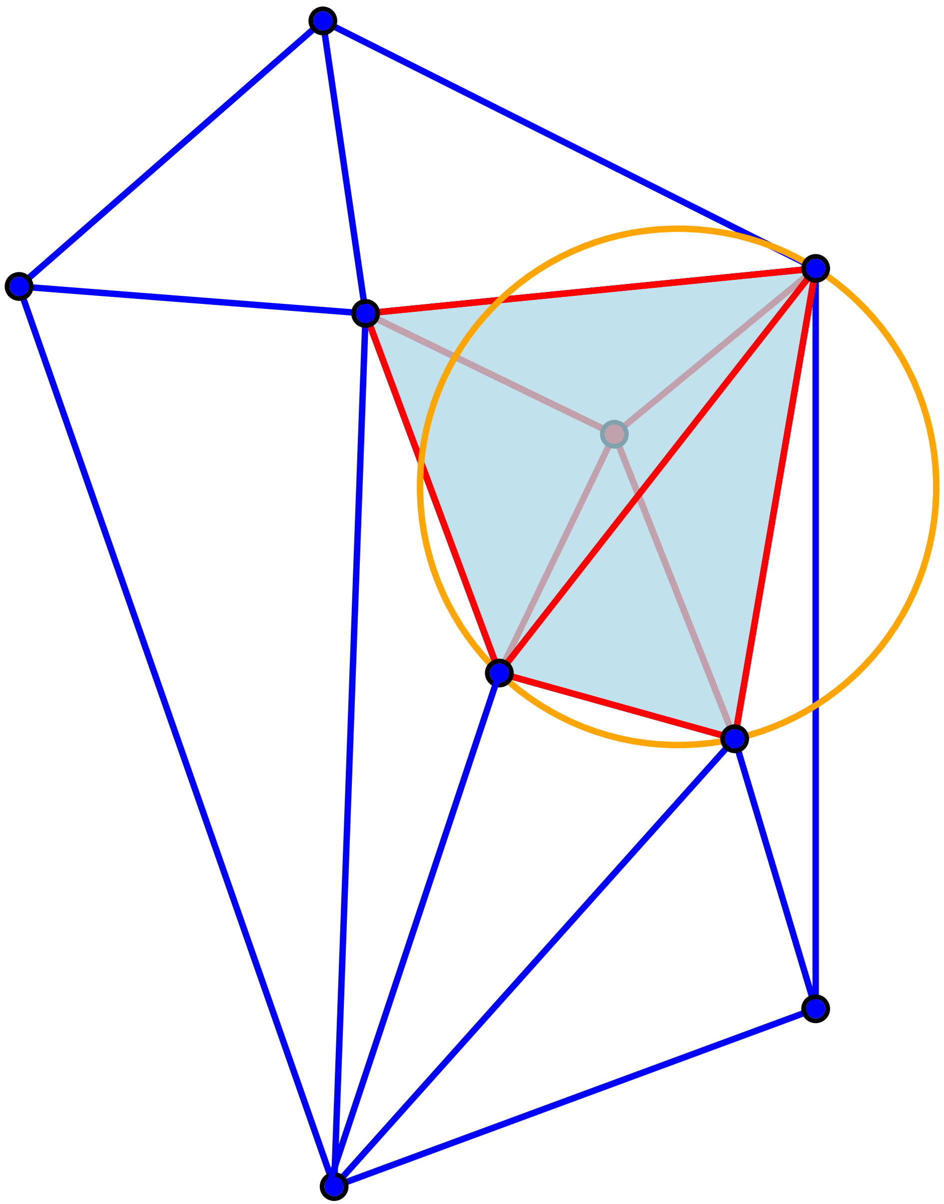
for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 4

only one predicate

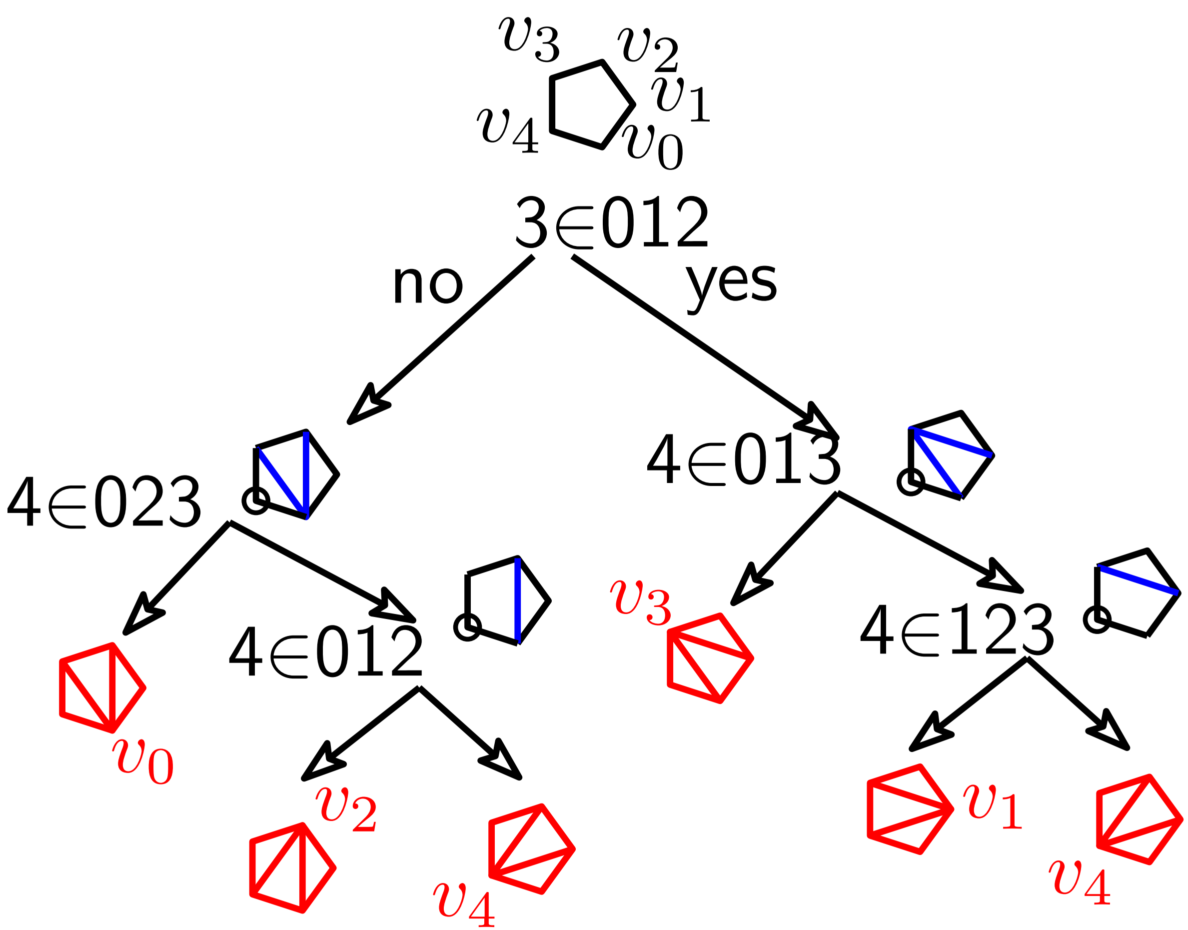


for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 5



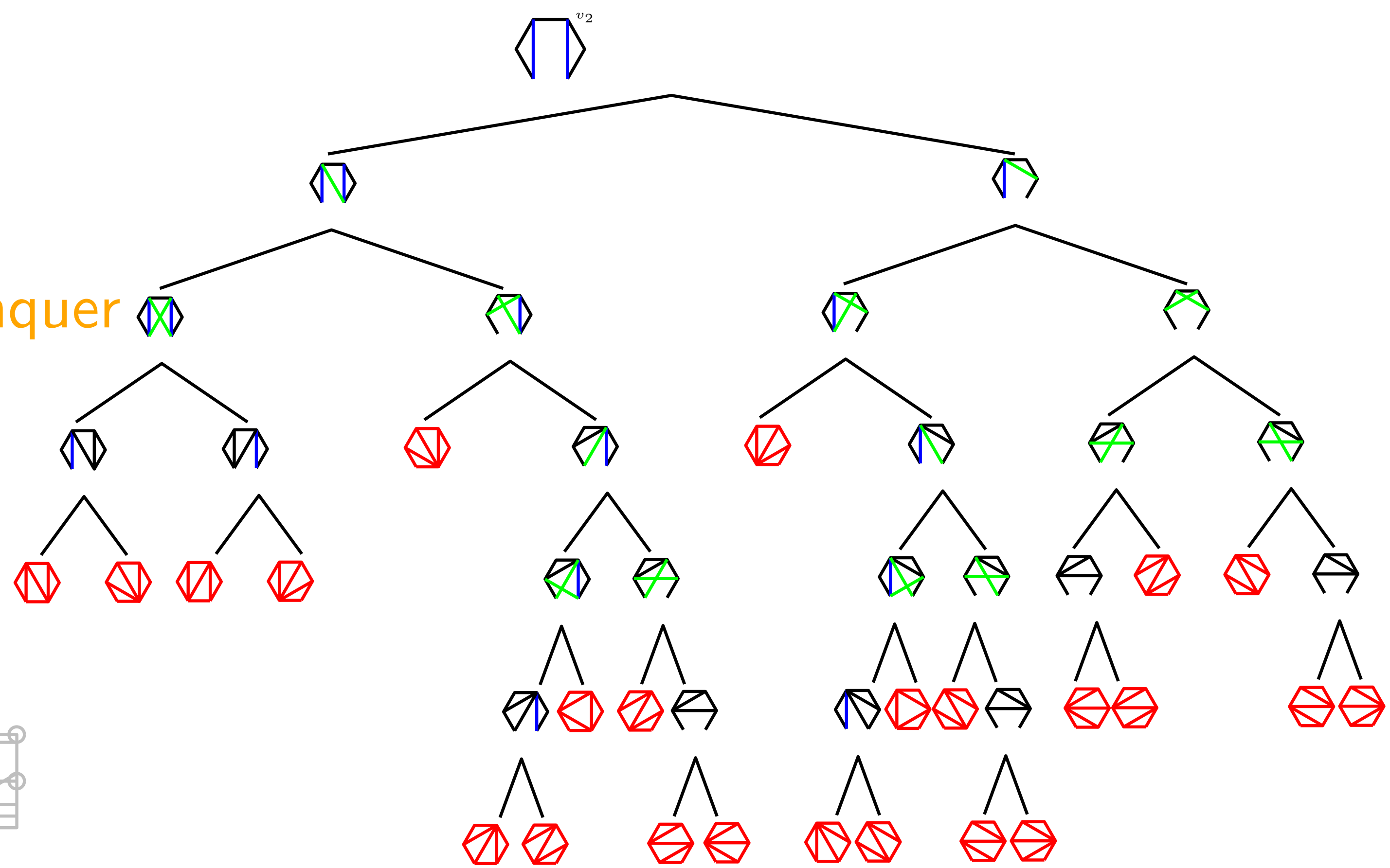
for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 6

"manual" d&conquer



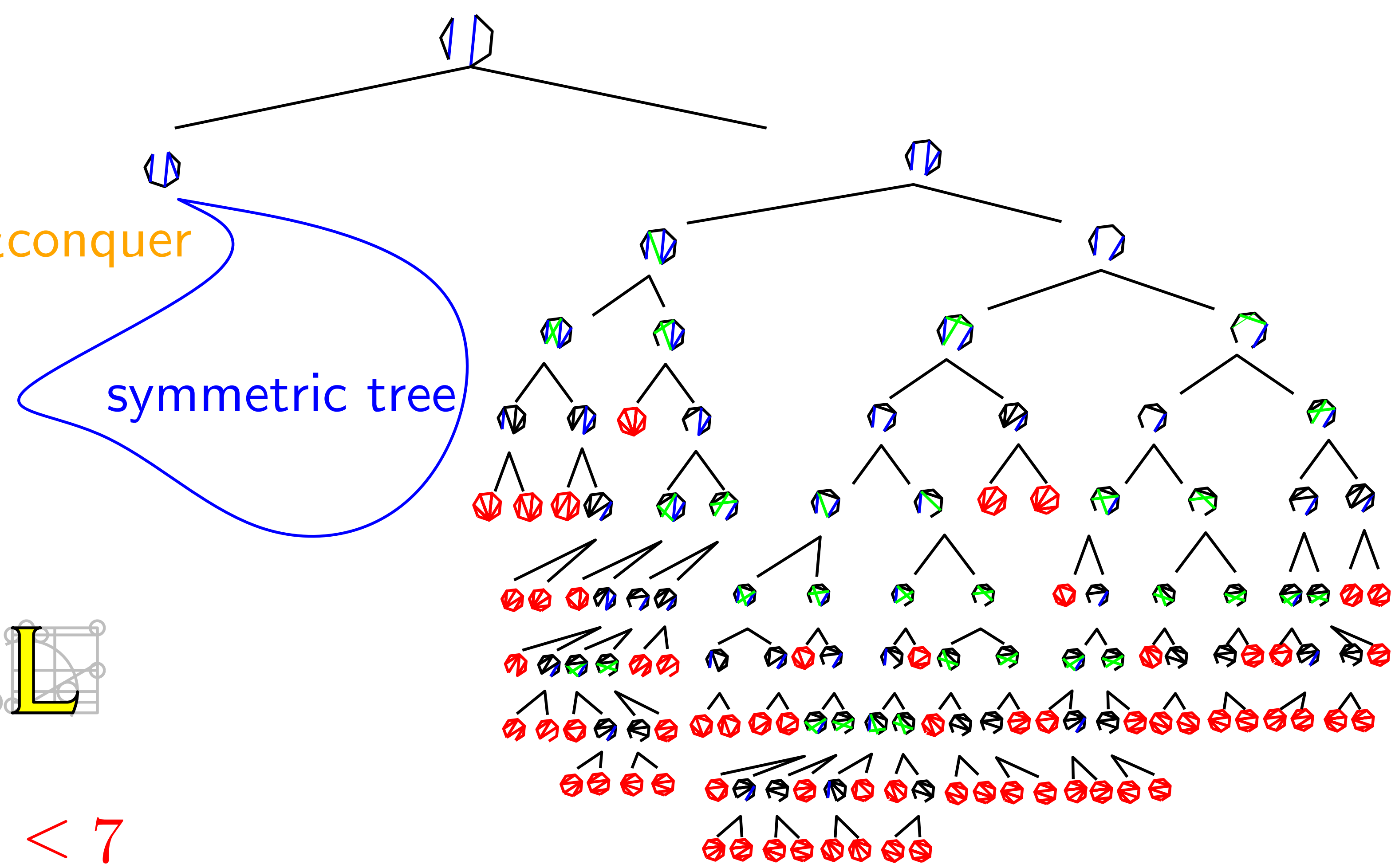
for degree ≤ 7

Delaunay Triangulation: deletion algorithm (sketch)

Decision tree for small holes

degree 7

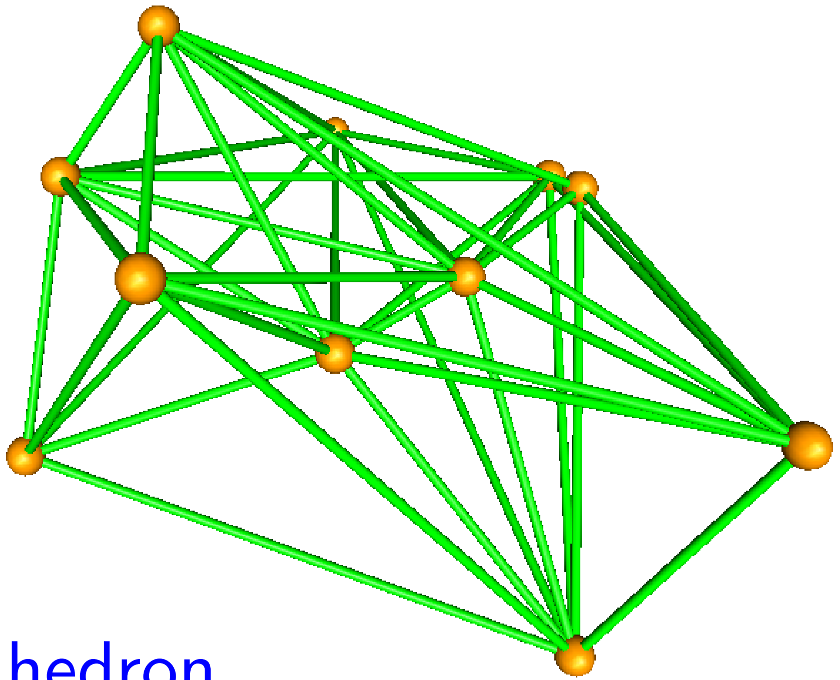
"manual" d&conquer



CGAL

for degree ≤ 7

Delaunay Triangulation: 3D



Same as 2D

Dual Voronoi diagram

Empty sphere property

Triangle $\longrightarrow$ Tetrahedron

Duality with 4D convex hull

Incremental algorithm (find the hole and star)

Delaunay

Convex hull

Higher dimensions

Dehn Sommerville relations $\qquad f_i = \sharp(\text{faces of dim } i)$

Euler: $\qquad f_0 - f_1 + f_2 - \dots f_{d-1} = (-1)^{d-1} + 1$

$$\sum_j = k^{d-1} - 1^j \left(\begin{array}{c} j+1 \\ k+1 \end{array} \right) f_j = (-1)^{d-1} f_k$$

$-1 \le k \le d-2 \qquad\qquad f_{-1} = f_d = 1$

$\left\lfloor \dfrac{d+1}{2} \right\rfloor$ independent equations

Same as 2D

Dual

Empty

Tri

Duality with 4D convex hull

Incremental algorithm (find the hole and star)

Delaunay

Same as 2D

Dual

Empty

Tri

Convex hull

Higher dimensions

Dehn Sommerville relations $\qquad f_i = \sharp(\text{faces of dim } i)$

Euler: $\qquad f_0 - f_1 + f_2 - \ldots f_{d-1} = (-1)^{d-1} + 1$

$$\sum_j = k^{d-1} - 1^j \begin{pmatrix} j+1 \\ k+1 \end{pmatrix} f_j = (-1)^{d-1} f_k$$

$-1 \leq k \leq d-2 \qquad f_{-1} = f_d = 1$

$\left\lfloor \frac{d+1}{2} \right\rfloor$ independent equations

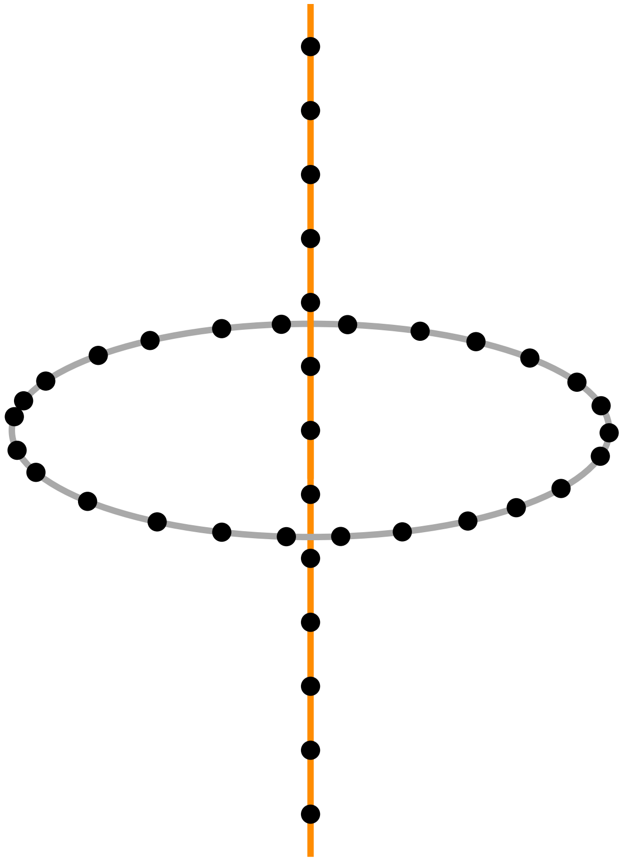
quadratic ?

Duality with 4D convex hull

Incremental algorithm (find the hole and star)

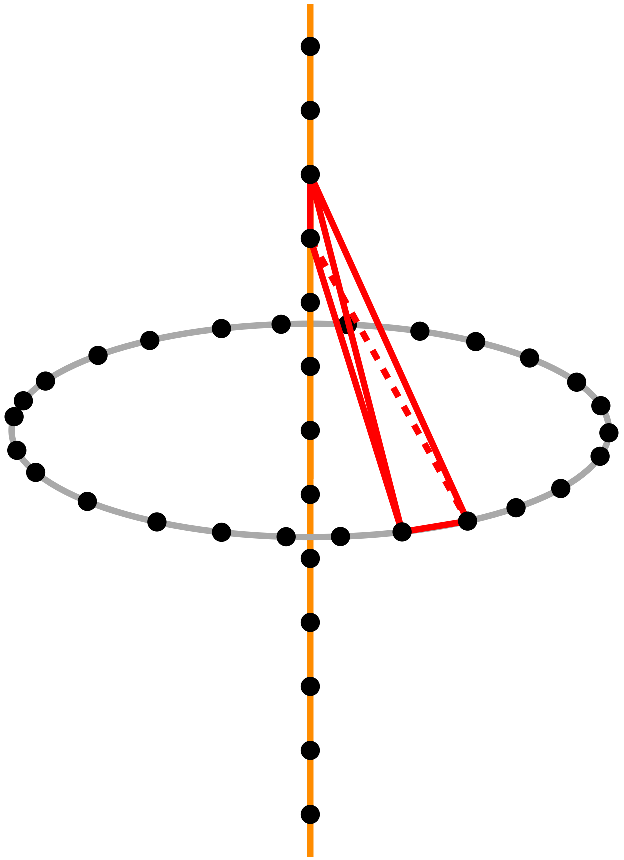
Delaunay Triangulation: 3D

Quadratic examples



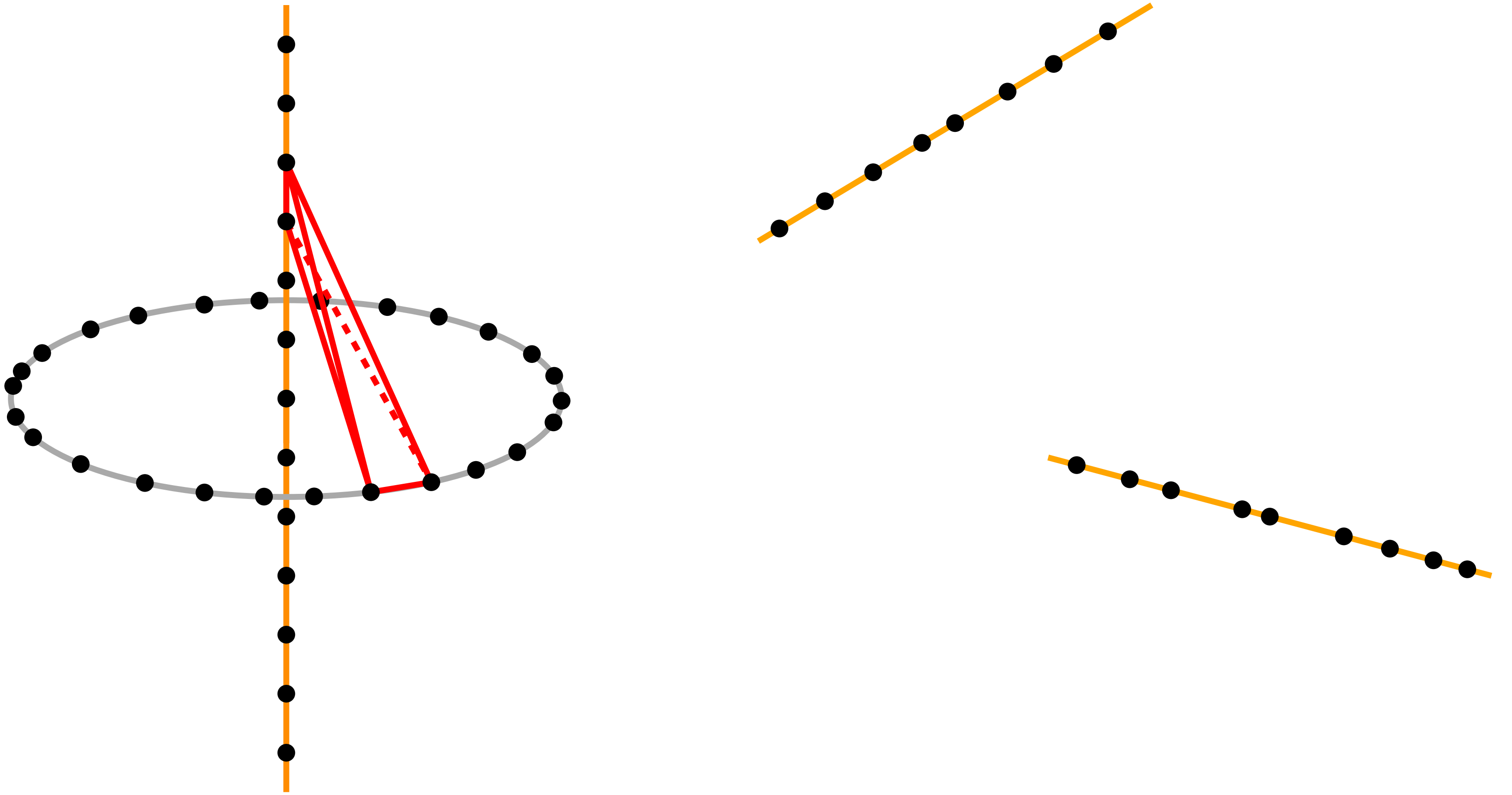
Delaunay Triangulation: 3D

Quadratic examples



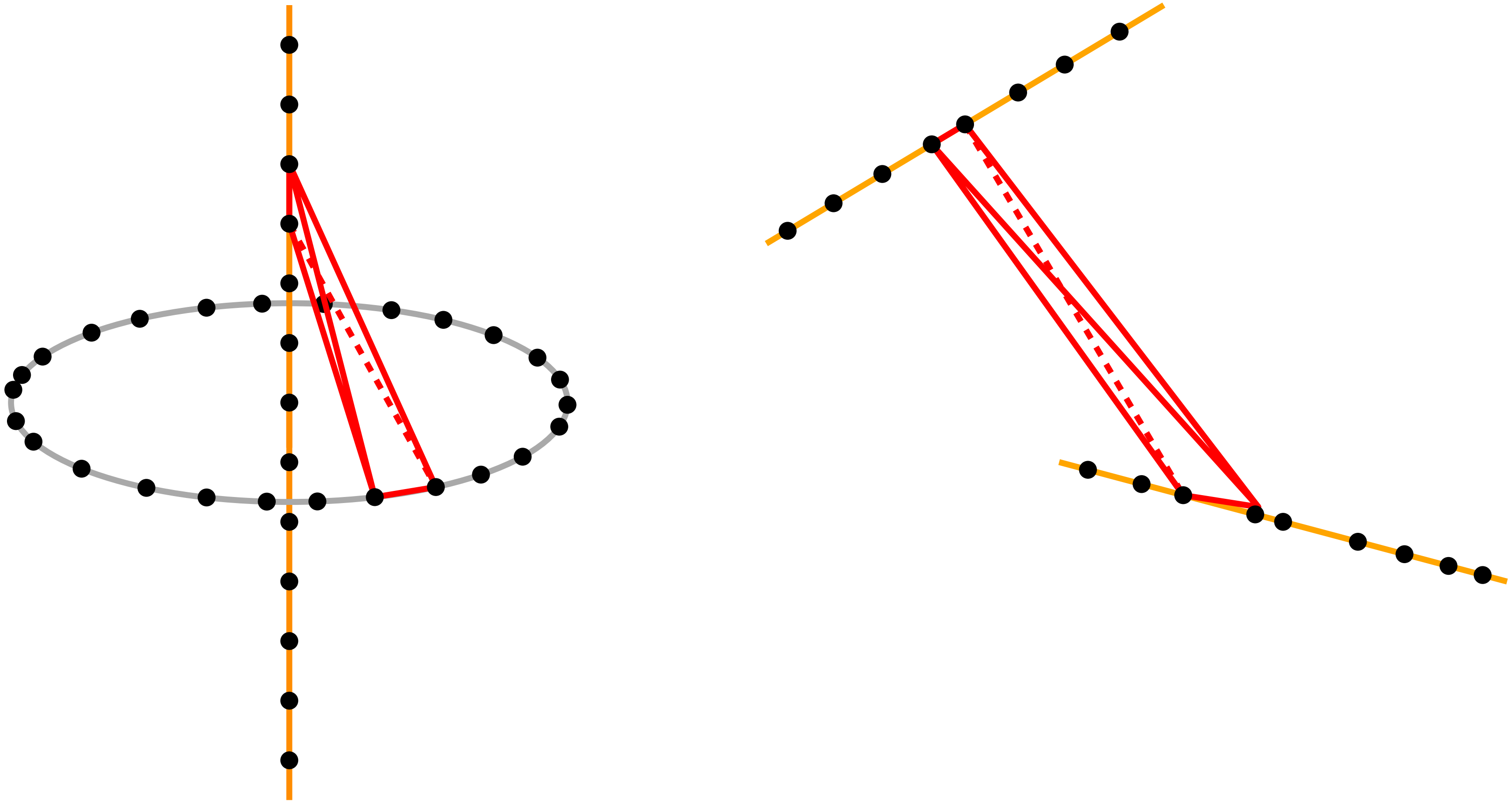
Delaunay Triangulation: 3D

Quadratic examples



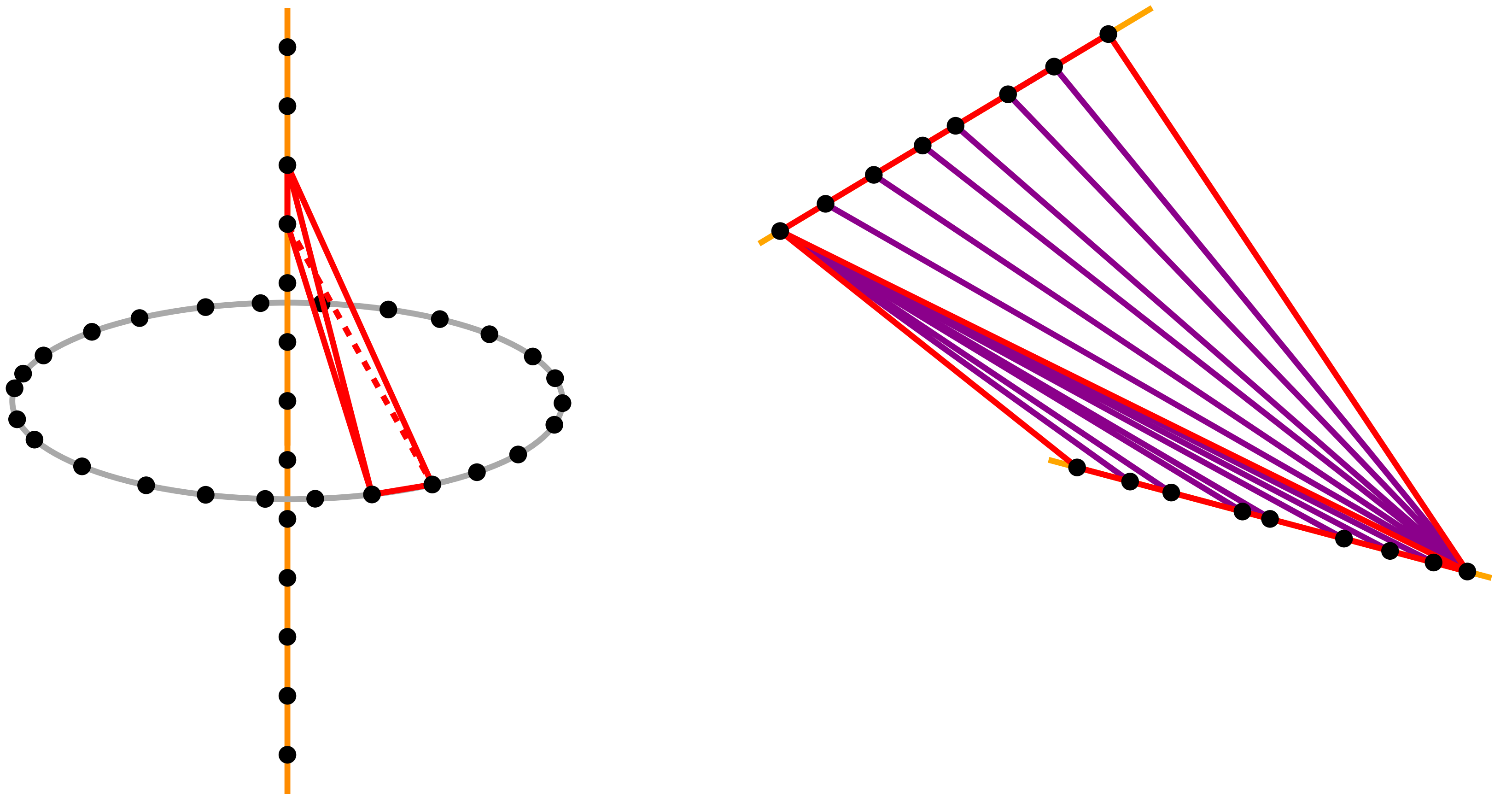
Delaunay Triangulation: 3D

Quadratic examples



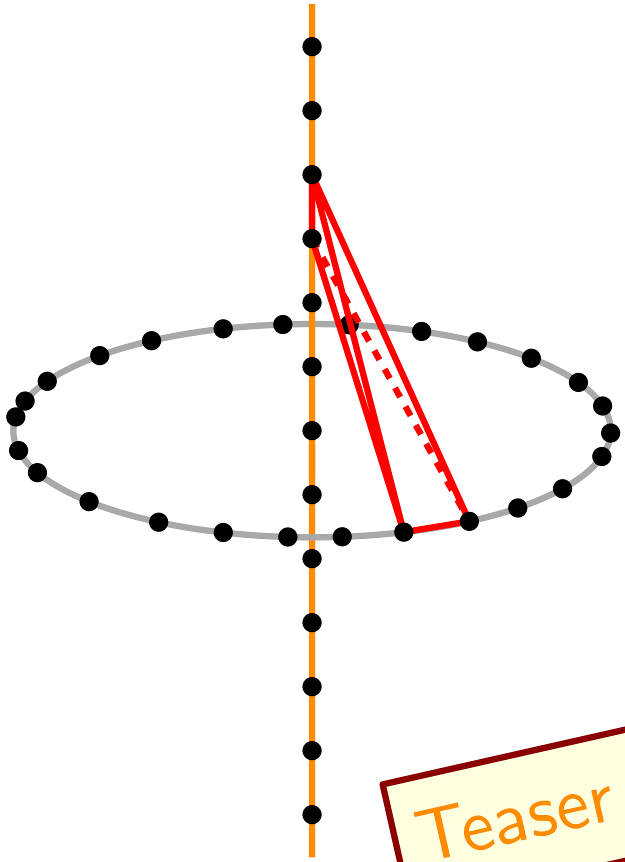
Delaunay Triangulation: 3D

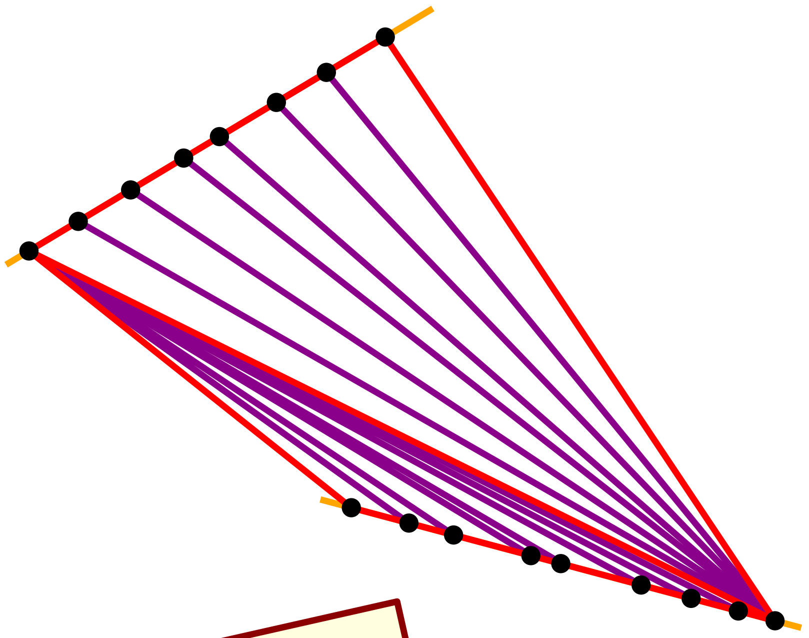
Quadratic examples



Delaunay Triangulation: 3D

Quadratic examples





Teaser probability lecture

Better results for random points

Delaunay Triangulation: 3D

Algorithms

4D convex hull duality

~~Flip~~

Incremental

Delaunay Triangulation: 3D

Algorithms

4D convex hull duality	$O(f \log n + n^{\frac{4}{3}})$ or $\Theta(n^2)$
~~Flip~~	
Incremental	$\Theta(n^3)$
	practical

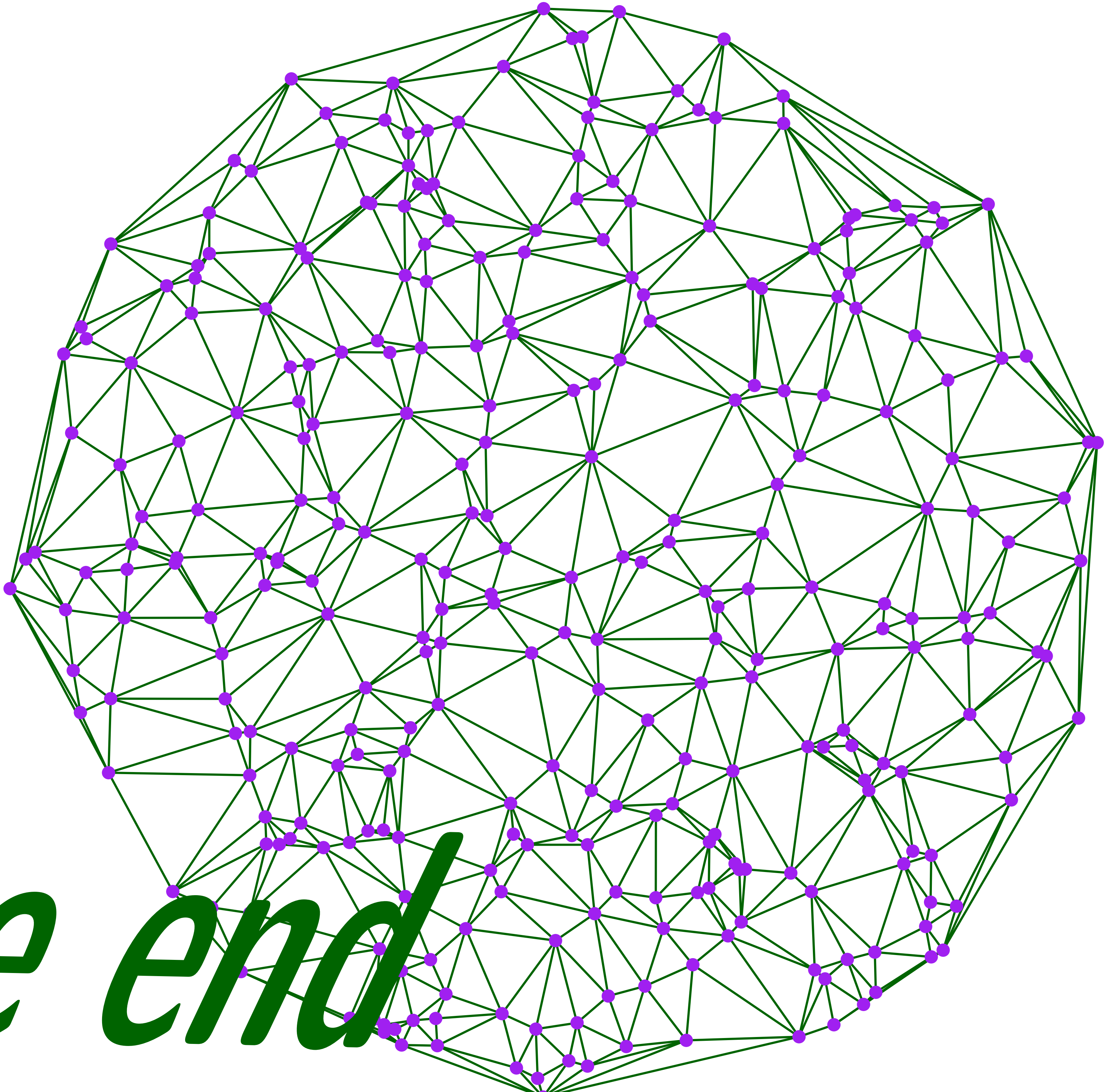
Teaser randomization lecture

Delaunay Triangulation: higher dimensions

$d+1$ convex hull duality $O\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$

Incremental practical $O(n)$ for random points

coeff exponential in d



The end