

# Semantics for Probabilistic and Quantum Effects

## (Unfinished Draft)

Xiaodong Jia<sup>\*‡</sup>, Bert Lindenhovius<sup>†</sup>, Michael Mislove<sup>‡</sup> and Vladimir Zamdzhiev<sup>§</sup>

<sup>\*</sup> School of Mathematics, Hunan University, Changsha, 410082, China

<sup>†</sup> Department of Knowledge-Based Mathematical Systems, Johannes Kepler Universität, Linz, Austria

<sup>‡</sup> Department of Computer Science, Tulane University, New Orleans, LA, USA

<sup>§</sup> Université de Lorraine, CNRS, Inria, LORIA, F 54000 Nancy, France

### Abstract

We consider a programming language that can manipulate both classical and quantum information. The classical fragment of the language is the Probabilistic FixPoint Calculus (PFPC) which is a lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice. The quantum fragment is a first-order programming language with inductive types that can manipulate quantum information. The two fragments are related by an interface which specifies how classical probabilistic effects are induced by quantum measurements, and vice-versa, it also specifies how classical (effectful) programs may influence the quantum dynamics. We also describe a sound and computationally adequate denotational semantics for the language. Classical probabilistic effects are interpreted using a recently-described commutative probabilistic monad on DCPO. Quantum effects and resources are interpreted in a category of von Neumann algebras which we show is enriched over continuous domains. This strong sense of enrichment allows us to develop novel semantic methods that we use to interpret the relationship between the quantum and classical probabilistic effects.

### I. INTRODUCTION

Variational quantum algorithms [1], [2] are becoming increasingly important in quantum computation. The main idea there is to use hybrid classical-quantum algorithms which work in tandem in order to solve certain computational problems. The classical part of the computation is executed on a classical computer and the quantum part on a quantum computer. In such a situation, intermediary results obtained from the quantum part of the computation are produced with certain probabilities, then passed onto the classical CPU which performs some computations that are used to determine how to tune the parameters of the quantum part of the algorithm, thereby influencing the quantum dynamics.

These kinds of hybrid classical-quantum algorithms pose interesting challenges for the design of suitable programming languages. Clearly, if we wish to understand how to program in such scenarios, one needs to devise a type system equipped with an operational semantics that correctly models the manipulation of *quantum resources*. On the other hand, quantum measurements induce *probabilistic computational effects* which are inherited by the classical side of the system. However, quantum information behaves very differently from classical information. For instance, quantum information cannot be copied [3]. In order to avoid such runtime errors, a *substructural type system* [4]–[6] where contraction is restricted is appropriate. On the other hand, when manipulating classical information, such restrictions are not only unnecessary, but in fact, very inconvenient.

#### A. Our Contributions

In this paper, we describe a programming language that is suitable for hybrid classical-quantum computation. From a type-theoretic perspective, our language can be understood as an extension of the linear/non-linear calculus [5], [6] with primitives for manipulating quantum resources. The language has two kinds of judgements: a classical (non-linear) judgement which represents classical programs and a quantum (linear) judgement which represents quantum programs. Our type system also contains hybrid classical-quantum formation rules which explain how classical and quantum computation interact with each other.

From an operational perspective, our language supports both classical probabilistic and quantum effects. The quantum dynamics are modelled via a probabilistic reduction relation on quantum configurations (terms with embedded quantum data within them) where the probabilities of reduction are determined in accordance with the laws of quantum mechanics. The classical dynamics are modelled via a probabilistic reduction relation on terms, where the probabilities of reduction are induced by the quantum dynamics.

We also provide a denotational interpretation of our system. We use our recently-described commutative probabilistic monad on the category DCPO in order to interpret the classical fragment of our language [7] (we also recall our construction here). We interpret quantum effects and resources in the category of hereditarily atomic von Neumann algebras which we show is categorically enriched over *continuous domains*. This is a very strong sense of enrichment which allows us to develop novel semantic methods that we use to interpret the relationship between the quantum and classical probabilistic effects. In particular, we can model the transition between classical and quantum computation using barycentre maps which are well-behaved under such enrichment.

## II. SYNTAX AND OPERATIONAL SEMANTICS

In this section we describe the syntax and operational semantics of our language. The classical fragment of our language is the *Probabilistic FixPoint Calculus* (PFPC). The presentation we choose for PFPC is based on FPC [8]–[10]. The same language is also considered by Jones [11], but with a slightly different syntax. The quantum fragment of the language is a first-order linear type system that can manipulate quantum information.

### A. The Type Structure

Recursive types in our language are formed in the same way as in FPC. We use  $X, Y$  to range over *type variables* and we use  $\Theta$  to range over *type contexts*. A type context  $\Theta = X_1, \dots, X_n$  is *well-formed*, written  $\Theta \vdash$ , if all type variables within it are distinct. We use  $A, B$  to range over the *types* of our language which are defined in Figure 2. We write  $\Theta \vdash A$  to indicate that type  $A$  is *well-formed* in type context  $\Theta$  whenever the judgement is derivable via the rules in Figure 2. A type  $A$  is *closed* when  $\cdot \vdash A$ . We remark that there are no restrictions on the admissible logical polarities of our type expressions, even when forming recursive types.

**Example 1.** Some important (closed) types may be defined in the following way:

- *Booleans* as  $\text{Bool} \stackrel{\text{def}}{=} 1 + 1$ ;
- *Natural numbers* as  $\text{Nat} \stackrel{\text{def}}{=} \mu X. 1 + X$ ;
- *Lists of type  $A$*  as  $\text{List}(A) \stackrel{\text{def}}{=} \mu X. 1 + A \times X$ ;
- *Streams of type  $A$*  as  $\text{Stream}(A) \stackrel{\text{def}}{=} \mu X. 1 \rightarrow A \times X$ ;

and many others.

### B. The Term Language

We now explain the syntax we use for terms. When forming terms and term contexts, we implicitly assume that all types within are closed and well-formed. We use  $x, y$  to range over *term variables* and we use  $\Phi, \Gamma$  to range over *term contexts*. A (well-formed) term context  $\Gamma = x_1 : A_1, \dots, x_n : A_n$  is a list of (distinct) variables with their types. The terms and the values are specified in Figure 1 and their formation rules in Figure 3 and Figure 4. The notation  $A[\mu X.A/X]$  indicates type substitution which is defined in the standard way. A term  $m$  of type  $A$  is *closed* when  $\cdot \vdash m : A$  and in this case we also simply write  $m : A$ .

**Example 2.** Important closed values include: the *false* and *true* values given by  $\text{ff} \stackrel{\text{def}}{=} \text{in}_1() : \text{Bool}$  and  $\text{tt} \stackrel{\text{def}}{=} \text{in}_2() : \text{Bool}$ ; the *zero natural number*  $\text{zero} \stackrel{\text{def}}{=} \text{fold in}_1() : \text{Nat}$  and the *successor function*  $\text{succ} \stackrel{\text{def}}{=} \lambda n^{\text{Nat}}. \text{fold in}_2 n : \text{Nat} \rightarrow \text{Nat}$ ; among many others.

### C. The Reduction Rules

To describe execution of programs, we use a small-step call-by-value operational semantics which is described in Figures 8–10. The reduction relation  $m \xrightarrow{p} n$  should be understood as specifying that term  $m$  reduces to term  $n$  with probability  $p \in [0, 1]$  in exactly one step. Reduction for the quantum fragment is described in terms of *quantum configurations*  $[\psi, \ell, \mathbf{q}]$ , which are terms with embedded quantum data. In such a configuration,  $\mathbf{x}_1 : \mathbf{qbit}, \dots, \mathbf{x}_n : \mathbf{qbit} \vdash \mathbf{q} : \mathbf{A}$  is a term,  $\psi \in \mathbb{C}^{2^n}$  is a quantum state,  $\ell$  is a bijection from the set of free variables of  $\mathbf{q}$  to  $\{1, \dots, n\}$  which is used to associate qubits of  $\psi$  with free variables of  $\mathbf{q}$ . See [12], [13] for more information.

**Assumption 3.** Throughout the rest of the paper, we implicitly assume that all types, terms and contexts are well-formed.

### D. Recursion and Asymptotic Behaviour of Reduction

We now explain how to compute the probability that a classical term reduces to another one in any number of steps.

We may determine the overall probability that a term  $m$  reduces to a value  $v$  in the same way as in [13]. The *probability weight* of a reduction path  $\pi = (m_1 \xrightarrow{p_1} \dots \xrightarrow{p_n} m_n)$  is  $P(\pi) \stackrel{\text{def}}{=} \prod_{i=1}^n p_i$ . The probability that term  $m$  reduces to the value  $v$  in *at most*  $n$  steps is

$$P(m \rightarrow_{\leq n} v) \stackrel{\text{def}}{=} \sum_{\pi \in \text{Paths}_{\leq n}(m, v)} P(\pi),$$

where  $\text{Paths}_{\leq n}(m, v)$  is the set of all reduction paths from  $m$  to  $v$  of length at most  $n$ . The probability that term  $m$  reduces to value  $v$  (in *any finite number* of steps) is  $P(m \rightarrow_* v) \stackrel{\text{def}}{=} \sup_i P(m \rightarrow_{\leq i} v)$ .

|                          |                          |                            |   |
|--------------------------|--------------------------|----------------------------|---|
| Quantum Type Variables   | $\mathbf{X}, \mathbf{Y}$ | Classical Type Variables   | $X, Y$  |
| Quantum Type Contexts    | $\Theta$                 | Classical Type Contexts    | $\Theta$  |
| Quantum Types            | $\mathbf{A}, \mathbf{B}$ | Classical Types            | $X \mid \mathbf{I} \mid P + R \mid P \times R \mid P \rightarrow R \mid Q(\mathbf{A}, \mathbf{B}) \mid \mu X.P$   |
| Observable Quantum Types | $\mathbf{O}$             | Observable Classical Types | $O$   |
| Quantum Term Variables   | $\mathbf{x}, \mathbf{y}$ | Classical Term Variables   | $x, y$  |
| Classical Terms          | $m, n$                   | Quantum Configurations     | $\mathcal{C} ::= [\psi, \ell, \mathbf{q}]$  |
| Quantum Terms            | $\mathbf{q}, \mathbf{r}$ | Classical Terms            | $x \mid () \mid (m, n) \mid \pi_1 m \mid \pi_2 m \mid in_1 m \mid in_2 m \mid (\text{case } m \text{ of } in_1 x \Rightarrow n_1 \mid in_2 y \Rightarrow n_2) \mid \text{fold } m \mid \text{unfold } m \mid \lambda x.m \mid mn \mid \text{qfun } (\mathbf{x}_1, \dots, \mathbf{x}_n) \Rightarrow \mathbf{q} \mid \text{new} \mid \text{meas} \mid U \mid \text{run } \mathcal{C}$ |
| Classical Values         | $v, w$                   | Quantum Values             | $\mathbf{v}, \mathbf{w}$  |
| Quantum Values           | $\mathbf{v}, \mathbf{w}$ | Classical Values           | $v, w$  |

Fig. 1. Grammars for types, contexts, terms and configurations.

$$\begin{array}{c}
\frac{\Theta \vdash}{\Theta \vdash \Theta_i} \quad \frac{\Theta \vdash}{\Theta \vdash \mathbf{I}} \quad \frac{\Theta \vdash}{\Theta \vdash \mathbf{qbit}} \quad \frac{\Theta \vdash \mathbf{A} \quad \Theta \vdash \mathbf{B}}{\Theta \vdash \mathbf{A} \star \mathbf{B}} \star \in \{\oplus, \otimes\} \quad \frac{\Theta, \mathbf{X} \vdash \mathbf{A}}{\Theta \vdash \mu \mathbf{X}. \mathbf{A}} \\
\frac{\Theta \vdash}{\Theta \vdash \Theta_i} \quad \frac{\Theta \vdash}{\Theta \vdash \mathbf{I}} \quad \frac{\Theta \vdash P \quad \Theta \vdash R}{\Theta \vdash P \star R} \star \in \{+, \times, \rightarrow\} \quad \frac{\cdot \vdash \mathbf{A} \quad \cdot \vdash \mathbf{B}}{\Theta \vdash Q(\mathbf{A}, \mathbf{B})} \quad \frac{\Theta \vdash}{\Theta \vdash \mu \mathbf{X}. P}
\end{array}$$

Fig. 2. Formation rules for types. Observable quantum/classical types are subsets of the quantum/classical types.

$$\begin{array}{c}
\frac{}{\Phi, x : P \vdash x : P} \quad \frac{}{\Phi \vdash () : \mathbf{I}} \quad \frac{\Phi \vdash m : P \quad \Phi \vdash n : R}{\Phi \vdash (m, n) : P \times R} \quad \frac{\Phi \vdash m : P_1 \times P_2}{\Phi \vdash \pi_i m : P_i} \quad i \in \{1, 2\} \\
\frac{\Phi \vdash m : P}{\Phi \vdash in_1 m : P + R} \quad \frac{\Phi \vdash m : R}{\Phi \vdash in_2 m : P + R} \quad \frac{\Phi \vdash m : P_1 + P_2 \quad \Phi, x : P_1 \vdash n_1 : R \quad \Phi, y : P_2 \vdash n_2 : R}{\Phi \vdash (\text{case } m \text{ of } in_1 x \Rightarrow n_1 \mid in_2 y \Rightarrow n_2) : R} \\
\frac{\Phi, x : P \vdash m : R}{\Phi \vdash \lambda x^P. m : P \rightarrow R} \quad \frac{\Phi \vdash m : P \rightarrow R \quad \Phi \vdash n : P}{\Phi \vdash mn : R} \quad \frac{\Phi \vdash m : P[\mu X.P/X]}{\Phi \vdash \text{fold } m : \mu X.P} \quad \frac{\Phi \vdash m : \mu X.P}{\Phi \vdash \text{unfold } m : P[\mu X.P/X]}
\end{array}$$

Fig. 3. Formation rules for terms in the classical fragment (FPC).

$$\begin{array}{c}
\frac{}{\Phi; \mathbf{x} : \mathbf{A} \vdash \mathbf{x} : \mathbf{A}} \quad \frac{}{\Phi; \cdot \vdash \ast : \mathbf{I}} \quad \frac{\Phi; \Gamma_1 \vdash \mathbf{q} : \mathbf{I} \quad \Phi; \Gamma_2 \vdash \mathbf{r} : \mathbf{A}}{\Phi; \Gamma_1, \Gamma_2 \vdash \mathbf{q} : \mathbf{r} : \mathbf{A}} \quad \frac{\Phi; \Gamma_1 \vdash \mathbf{q} : \mathbf{A} \quad \Phi; \Gamma_2 \vdash \mathbf{r} : \mathbf{B}}{\Phi; \Gamma_1, \Gamma_2 \vdash \mathbf{q} \otimes \mathbf{r} : \mathbf{A} \otimes \mathbf{B}} \\
\frac{\Phi; \Gamma_1 \vdash \mathbf{q} : \mathbf{A}_1 \otimes \mathbf{A}_2 \quad \Phi; \Gamma_2, \mathbf{x} : \mathbf{A}_1, \mathbf{y} : \mathbf{A}_2 \vdash \mathbf{r} : \mathbf{B}}{\Phi; \Gamma_1, \Gamma_2 \vdash \text{let } \mathbf{x} \otimes \mathbf{y} = \mathbf{q} \text{ in } \mathbf{r} : \mathbf{B}} \quad \frac{\Phi; \Gamma \vdash \mathbf{q} : \mathbf{A}[\mu \mathbf{X}. \mathbf{A}/\mathbf{X}]}{\Phi; \Gamma \vdash \text{fold } \mathbf{q} : \mu \mathbf{X}. \mathbf{A}} \quad \frac{\Phi; \Gamma \vdash \mathbf{q} : \mu \mathbf{X}. \mathbf{A}}{\Phi; \Gamma \vdash \text{unfold } \mathbf{q} : \mathbf{A}[\mu \mathbf{X}. \mathbf{A}/\mathbf{X}]} \\
\frac{\Phi; \Gamma \vdash \mathbf{q} : \mathbf{A}}{\Phi; \Gamma \vdash \text{in}_1 \mathbf{q} : \mathbf{A} \oplus \mathbf{B}} \quad \frac{\Phi; \Gamma \vdash \mathbf{q} : \mathbf{B}}{\Phi; \Gamma \vdash \text{in}_2 \mathbf{q} : \mathbf{A} \oplus \mathbf{B}} \quad \frac{\Phi; \Gamma_1 \vdash \mathbf{q} : \mathbf{A}_1 \oplus \mathbf{A}_2 \quad \Phi; \Gamma_2, \mathbf{x} : \mathbf{A}_1 \vdash \mathbf{r}_1 : \mathbf{B} \quad \Phi; \Gamma_2, \mathbf{y} : \mathbf{A}_2 \vdash \mathbf{r}_2 : \mathbf{B}}{\Phi; \Gamma_1, \Gamma_2 \vdash (\text{case } \mathbf{q} \text{ of } \text{in}_1 \mathbf{x} \Rightarrow \mathbf{r}_1 \mid \text{in}_2 \mathbf{y} \Rightarrow \mathbf{r}_2) : \mathbf{B}}
\end{array}$$

Fig. 4. Formation rules for terms in the quantum fragment (first-order linear type system with inductive types).

$$\frac{}{\Phi \vdash \text{new} : Q(\mathbf{bit}, \mathbf{qbit})} \quad \frac{}{\Phi \vdash \text{meas} : Q(\mathbf{qbit}, \mathbf{bit})} \quad \frac{U \text{ is a unitary of arity of } n}{\Phi \vdash U : Q(\mathbf{qbit}^{\otimes n}, \mathbf{qbit}^{\otimes n})} \quad \mathbf{bit} \equiv \mathbf{I} \oplus \mathbf{I}$$

Fig. 5. Formation rules for terms that manipulate quantum information.

$$\begin{array}{c}
|\mathbf{X}| \stackrel{\text{def}}{=} X \quad |\mathbf{I}| \stackrel{\text{def}}{=} 1 \quad |\mathbf{O}_1 \otimes \mathbf{O}_2| \stackrel{\text{def}}{=} |\mathbf{O}_1| \times |\mathbf{O}_2| \quad |\mathbf{O}_1 \oplus \mathbf{O}_2| \stackrel{\text{def}}{=} |\mathbf{O}_1| + |\mathbf{O}_2| \quad |\mu \mathbf{X}. \mathbf{O}| \stackrel{\text{def}}{=} \mu X. |\mathbf{O}| \\
|\ast| \stackrel{\text{def}}{=} () \quad |\mathbf{v} \otimes \mathbf{w}| \stackrel{\text{def}}{=} (|\mathbf{v}|, |\mathbf{w}|) \quad |\text{in}_1 \mathbf{v}| \stackrel{\text{def}}{=} in_1 |\mathbf{v}| \quad |\text{in}_2 \mathbf{v}| \stackrel{\text{def}}{=} in_2 |\mathbf{v}| \quad |\text{fold } \mathbf{v}| \stackrel{\text{def}}{=} \text{fold } |\mathbf{v}|
\end{array}$$

Fig. 6. Translation between observable quantum types/values and observable classical types/values.

$$\begin{array}{c}
\frac{\Phi; \mathbf{x}_1 : \mathbf{A}_1, \dots, \mathbf{x}_n : \mathbf{A}_n \vdash \mathbf{q} : \mathbf{B}}{\Phi \vdash \text{qfun } (\mathbf{x}_1, \dots, \mathbf{x}_n) \Rightarrow \mathbf{q} : Q(\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_n, \mathbf{B})} \quad \frac{\Phi \vdash m : Q(\mathbf{A}, \mathbf{B}) \quad \Phi; \Gamma \vdash \mathbf{q} : \mathbf{A}}{\Phi; \Gamma \vdash m \mathbf{q} : \mathbf{B}} \quad \frac{\Phi \vdash \mathcal{C} : \mathbf{O} \quad \mathcal{C} \text{ is total}}{\Phi \vdash \text{run } \mathcal{C} : |\mathbf{O}|} \\
\frac{\Phi \vdash m : |\mathbf{O}|}{\Phi; \cdot \vdash \text{init } m : \mathbf{O}} \quad \frac{\Phi; \Gamma_1 \vdash \mathbf{q} : \mathbf{O} \quad \Phi, x : |\mathbf{O}|; \Gamma_2 \vdash \mathbf{r} : \mathbf{A}}{\Phi; \Gamma_1, \Gamma_2 \vdash \text{let } x = \text{lift } \mathbf{q} \text{ in } \mathbf{r} : \mathbf{A}}
\end{array}$$

Fig. 7. Formation rules for terms that mediate between the quantum and classical modes of operation.

$$\begin{array}{l}
\pi_1(v, w) \xrightarrow{1} v \quad \pi_2(v, w) \xrightarrow{1} w \quad [\psi, \ell, \mathbf{let} \mathbf{x} \otimes \mathbf{y} = \mathbf{v} \otimes \mathbf{w} \mathbf{in} \mathbf{r}] \xrightarrow{1} [\psi, \ell, \mathbf{r}[\mathbf{v}/\mathbf{x}, \mathbf{w}/\mathbf{y}]] \\
(\mathbf{case} \mathbf{in}_1 v \mathbf{of} \mathbf{in}_1 x \Rightarrow n_1 \mid \mathbf{in}_2 y \Rightarrow n_2) \xrightarrow{1} n_1[v/x] \quad [\psi, \ell, \mathbf{case} \mathbf{in}_1 \mathbf{v} \mathbf{of} \mathbf{in}_1 \mathbf{x} \Rightarrow \mathbf{r}_1 \mid \mathbf{in}_2 \mathbf{y} \Rightarrow \mathbf{r}_2] \xrightarrow{1} [\psi, \ell, \mathbf{r}_1[\mathbf{v}/\mathbf{x}]] \\
(\mathbf{case} \mathbf{in}_2 v \mathbf{of} \mathbf{in}_1 x \Rightarrow n_1 \mid \mathbf{in}_2 y \Rightarrow n_2) \xrightarrow{1} n_2[v/y] \quad [\psi, \ell, \mathbf{case} \mathbf{in}_2 \mathbf{v} \mathbf{of} \mathbf{in}_1 \mathbf{x} \Rightarrow \mathbf{r}_1 \mid \mathbf{in}_2 \mathbf{y} \Rightarrow \mathbf{r}_2] \xrightarrow{1} [\psi, \ell, \mathbf{r}_2[\mathbf{v}/\mathbf{y}]] \\
\mathbf{unfold} \mathbf{fold} v \xrightarrow{1} v \quad [\psi, \ell, \mathbf{unfold} \mathbf{fold} \mathbf{v}] \xrightarrow{1} [\psi, \ell, \mathbf{v}] \\
(\lambda x. m)v \xrightarrow{1} m[v/x] \quad [\psi, \ell, *; \mathbf{r}] \xrightarrow{1} [\psi, \ell, \mathbf{r}] \quad [\psi, \ell, \mathbf{let} \mathbf{x} = \mathbf{lift} \mathbf{v} \mathbf{in} \mathbf{r}] \xrightarrow{1} [\psi, \ell, \mathbf{r}[\mathbf{v}/\mathbf{x}]] \\
E ::= [\cdot] \mid (E, m) \mid (v, E) \mid \pi_i E \mid \mathbf{in}_i E \mid (\mathbf{case} E \mathbf{of} \mathbf{in}_1 x \Rightarrow n_1 \mid \mathbf{in}_2 y \Rightarrow n_2) \mid Em \mid vE \mid (\mathbf{un})\mathbf{fold} E \\
\mathbf{E} ::= [\cdot] \mid \mathbf{E} \otimes \mathbf{q} \mid \mathbf{v} \otimes \mathbf{E} \mid \mathbf{let} \mathbf{x} \otimes \mathbf{y} = \mathbf{E} \mathbf{in} \mathbf{r} \mid \mathbf{in}_i \mathbf{E} \mid (\mathbf{case} \mathbf{E} \mathbf{of} \mathbf{in}_1 \mathbf{x} \Rightarrow \mathbf{r}_1 \mid \mathbf{in}_2 \mathbf{y} \Rightarrow \mathbf{r}_2) \mid \mathbf{E}; \mathbf{q} \mid (\mathbf{un})\mathbf{fold} \mathbf{E} \mid \mathbf{let} x = \mathbf{lift} \mathbf{E} \mathbf{in} \mathbf{r} \\
\frac{m \xrightarrow{p} m'}{E[m] \xrightarrow{p} E[m']} \quad \frac{[\psi, \ell, \mathbf{q}] \xrightarrow{p} [\psi', \ell', \mathbf{q}'] \quad \ell \cap \ell_0 = \emptyset = \ell' \cap \ell_0}{[\psi, \ell \cup \ell_0, \mathbf{E}[\mathbf{q}]] \xrightarrow{p} [\psi', \ell' \cup \ell_0, \mathbf{E}[\mathbf{q}']]}
\end{array}$$

Fig. 8. Grammars for classical evaluation contexts ( $E$ ), quantum evaluation contexts ( $\mathbf{E}$ ) and associated reduction rules.

$$[\psi, \ell, U(\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_k)] \xrightarrow{1} [\sigma \circ (U \otimes \text{id}) \circ \sigma^{-1}, \ell, \mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_k], \text{ for any permutation } \sigma, \text{ s.t. } \sigma(i) = \ell(\mathbf{x}_i) \text{ for } 1 \leq i \leq k.$$

$$[\psi, \emptyset, \mathbf{new} \mathbf{ff}] \xrightarrow{1} [\psi \otimes |0\rangle, \{\mathbf{x} \mapsto \dim(\psi) + 1\}, \mathbf{x}] \quad [\psi, \emptyset, \mathbf{new} \mathbf{tt}] \xrightarrow{1} [\psi \otimes |1\rangle, \{\mathbf{x} \mapsto \dim(\psi) + 1\}, \mathbf{x}] \quad \text{with } \mathbf{x} \text{ chosen fresh.}$$

$$\begin{array}{l}
\left[ \alpha \left( \sum_i \alpha_i |b_i\rangle \otimes |0\rangle \otimes |b'_i\rangle \right) + \beta \left( \sum_i \beta_i |b_i\rangle \otimes |1\rangle \otimes |b'_i\rangle \right), \{\mathbf{y} \mapsto j\}, \mathbf{meas} \mathbf{y} \right] \xrightarrow{|\alpha|^2} \left[ \sum_i \alpha_i |b_i\rangle \otimes |b'_i\rangle, \emptyset, \mathbf{ff} \right] \\
\left[ \alpha \left( \sum_i \alpha_i |b_i\rangle \otimes |0\rangle \otimes |b'_i\rangle \right) + \beta \left( \sum_i \beta_i |b_i\rangle \otimes |1\rangle \otimes |b'_i\rangle \right), \{\mathbf{y} \mapsto j\}, \mathbf{meas} \mathbf{y} \right] \xrightarrow{|\beta|^2} \left[ \sum_i \beta_i |b_i\rangle \otimes |b'_i\rangle, \emptyset, \mathbf{tt} \right]
\end{array}$$

Fig. 9. Rules for manipulating quantum information. In the rules for measurement,  $\dim(|b_i\rangle) = j - 1$ , so that the  $j$ -th qubit is measured.

$$\begin{array}{l}
[\psi, \ell, (\mathbf{qfun} (\mathbf{x}_1, \dots, \mathbf{x}_n) \Rightarrow \mathbf{q})(\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_n)] \xrightarrow{1} [\psi, \ell, \mathbf{q}[\mathbf{v}_1/\mathbf{x}_1, \dots, \mathbf{v}_n/\mathbf{x}_n]] \\
\mathbf{run} [1, \emptyset, \mathbf{v}] \xrightarrow{1} |\mathbf{v}\rangle \\
[\psi, \emptyset, \mathbf{init} |\mathbf{v}\rangle] \xrightarrow{1} [\psi, \emptyset, \mathbf{v}] \\
\frac{m \xrightarrow{p} m'}{[\psi, \ell, m\mathbf{q}] \xrightarrow{p} [\psi, \ell, m'\mathbf{q}]} \quad \frac{[\psi, \ell, \mathbf{q}] \xrightarrow{p} [\psi', \ell', \mathbf{q}']}{[\psi, \ell, v\mathbf{q}] \xrightarrow{p} [\psi', \ell', v\mathbf{q}']} \quad \frac{[\psi, \ell, \mathbf{q}] \xrightarrow{p} [\psi', \ell', \mathbf{q}']}{\mathbf{run} [\psi, \ell, \mathbf{q}] \xrightarrow{p} \mathbf{run} [\psi', \ell', \mathbf{q}']} \quad \frac{m \xrightarrow{p} m'}{[1, \emptyset, \mathbf{init} m] \xrightarrow{p} [1, \emptyset, \mathbf{init} m']}
\end{array}$$

Fig. 10. Rules for quantum function application, running quantum programs and initialising observable quantum data.

Finally, the probability that term  $m$  terminates is denoted  $\text{Halt}(m)$  and it is determined in the following way:

$$\text{Val}(m) \stackrel{\text{def}}{=} \{v \mid v \text{ is a value and } P(m \rightarrow_* v) > 0\} \quad (1)$$

$$\text{Halt}(m) \stackrel{\text{def}}{=} \sum_{v \in \text{Val}(m)} P(m \rightarrow_* v). \quad (2)$$

Note that the sum in (2) is countably infinite, in general.

The probability that a quantum configuration  $\mathcal{C}$  reduces to a value configuration  $\mathcal{V}$  (in any number of steps) is denoted  $P(\mathcal{C} \rightarrow_* \mathcal{V})$  and it is defined in the same way as for classical reductions by replacing the terms  $m$  and  $v$  with  $\mathcal{C}$  and  $\mathcal{V}$ .

### III. RECURSIVE TYPES AND PROBABILISTIC EFFECTS

In this section we consider the classical fragment of our language (PFPC) and we describe the mathematical structures we use to interpret it. From a type-theoretic perspective, the classical fragment is a call-by-value non-linear language that we model using a Kleisli adjunction, following Moggi [14]. The recursive type structure of the classical fragment induces term-level recursion, both of which we model following [15], [16] which uses domain-theoretic methods [17]. The classical fragment also exhibits probabilistic effects (induced by the quantum ones) that we interpret in the Kleisli category of a new commutative valuations monad over  $\mathbf{DCPO}$ , as described below.

### A. Domain-theoretic and Topological Preliminaries

A nonempty subset  $A$  of a partially ordered set (*poset*)  $D$  is *directed* if each pair of elements in  $A$  has an upper bound in  $A$ . A *directed-complete partial order*, (*dcpo*, for short) is a poset in which every directed subset  $A$  has a supremum  $\sup A$ . For example, the unit interval  $[0, 1]$  is a dcpo in the usual ordering. A function  $f: D \rightarrow E$  between two (posets) dcpo's is *Scott-continuous* if it is monotone and preserves (existing) suprema of directed subsets.

The category **DCPO** of dcpo's and Scott-continuous functions is complete, cocomplete and cartesian closed [17]. We denote with  $A_1 \times A_2$  ( $A_1 + A_2$ ) the categorical (co)product of the dcpo's  $A_1$  and  $A_2$  and with  $\pi_1, \pi_2$  ( $in_1, in_2$ ) the associated (co)projections. We denote with  $\emptyset$  and  $1$  the initial and terminal objects of **DCPO**; these are the empty dcpo and the singleton dcpo, respectively. **DCPO** is Cartesian closed, where the internal hom of  $A$  and  $B$  is  $[A \rightarrow B]$ , the Scott-continuous functions  $f: A \rightarrow B$  ordered pointwise.

The category **DCPO** $_{\perp!}$  of *pointed* dcpo's and *strict* Scott-continuous functions also is important. **DCPO** $_{\perp!}$  is symmetric monoidal closed when equipped with the smash product and strict Scott-continuous function space, and it is also complete and cocomplete [17].

The *Scott topology*  $\sigma D$  on a dcpo  $D$  consists of the upper subsets  $U = \uparrow U = \{x \in D \mid (\exists u \in U) u \leq x\}$  that are *inaccessible by directed suprema*: i.e., if  $A \subseteq D$  is directed and  $\sup A \in U$ , then  $A \cap U \neq \emptyset$ . The space  $(D, \sigma D)$  is also written as  $\Sigma D$ . Scott-continuous functions between dcpo's  $D$  and  $E$  are exactly the continuous functions between  $\Sigma D$  and  $\Sigma E$  [18, Proposition II-2.1]. We always equip  $[0, 1]$  with the Scott topology unless stated otherwise.

A subset  $B$  of a dcpo  $D$  is a *sub-dcpo* if every directed subset  $A \subseteq B$  satisfies  $\sup_D A \in B$ . In this case,  $B$  is a dcpo in the induced order from  $D$ . The *d-topology* on  $D$  is the topology whose closed subsets consist of sub-dcpo's of  $D$ . Open (closed) sets in the d-topology will be called *d-open* (*d-closed*). The *d-closure* of  $C \subseteq D$  is the topological closure of  $C$  with respect to the d-topology on  $D$ , which is the intersection of all sub-dcpo's of  $D$  containing  $C$ .

The family of open sets of a topological space  $X$ , denoted  $\mathcal{O}X$ , is a complete lattice in the inclusion order. The *specialization order*  $\leq_X$  on  $X$  is defined as  $x \leq_X y$  if and only if  $x$  is in the closure of  $\{y\}$ , for  $x, y \in X$ . We write  $\Omega X$  to denote  $X$  equipped with the specialization order. It is well-known that  $X$  is  $T_0$  if and only if  $\Omega X$  is a poset. A subset of  $X$  is called *saturated* if it is an upper set in  $\Omega X$ . A space  $X$  is called a *d-space* or a *monotone-convergence space* if  $\Omega X$  is a dcpo and each open set of  $X$  is Scott open in  $\Omega X$ . As an example,  $\Sigma D$  is always a d-space for each dcpo  $D$ . The full subcategory of **T** $_0$  consisting of d-spaces is denoted by **D**. There is a functor  $\Sigma: \mathbf{DCPO} \rightarrow \mathbf{D}$  that assigns the space  $\Sigma D$  to each dcpo  $D$ , and the map  $f: \Sigma D \rightarrow \Sigma E$  to the Scott-continuous map  $f: D \rightarrow E$ . Dually, the functor  $\Omega: \mathbf{D} \rightarrow \mathbf{DCPO}$  assigns  $\Omega X$  to each d-space  $X$  and the map  $f: \Omega X \rightarrow \Omega Y$  to each continuous map  $f: X \rightarrow Y$ . In fact,  $\Sigma \dashv \Omega$ , i.e.,  $\Sigma$  is left adjoint to  $\Omega$  [19].

A  $T_0$  space  $X$  is called *sober* if every nonempty closed irreducible subset of  $X$  is the closure of some (unique) singleton set, where  $A \subseteq X$  is *irreducible* if  $A \subseteq B \cup C$  with  $B$  and  $C$  nonempty closed subsets implies  $A \subseteq B$  or  $A \subseteq C$ . The category of sober spaces and continuous functions is denoted by **SOB**. Sober spaces are d-spaces, hence **SOB**  $\subseteq$  **D** [20].

### B. A Commutative Monad for Probability

To begin, a *subprobability valuation* on a topological space  $X$  is a Scott-continuous function  $\nu: \mathcal{O}X \rightarrow [0, 1]$  that is strict ( $\nu(\emptyset) = 0$ ), and modular ( $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$ ). The set of subprobability valuations on  $X$  is denoted by  $\mathcal{V}X$ . The *stochastic order* on  $\mathcal{V}X$  is defined pointwise:  $\nu_1 \leq \nu_2$  if and only if  $\nu_1(U) \leq \nu_2(U)$  for all  $U \in \mathcal{O}X$ .  $\mathcal{V}X$  is a pointed dcpo in the stochastic order, with least element given by the constantly zero valuation  $\mathbf{0}_X$  and where the supremum of a directed family  $\{\nu_i\}_{i \in I}$  is  $\sup_{i \in I} \nu_i \stackrel{\text{def}}{=} \lambda U. \sup_{i \in I} \nu_i(U)$ .

The canonical examples of subprobability valuations are the *Dirac valuations*  $\delta_x$  for  $x \in X$ , defined by  $\delta_x(U) = 1$  if  $x \in U$  and  $\delta_x(U) = 0$  otherwise.  $\mathcal{V}X$  enjoys a convex structure: if  $\nu_i \in \mathcal{V}X$  and  $r_i \geq 0$ , with  $\sum_{i=1}^n r_i \leq 1$ , then the convex sum  $\sum_{i=1}^n r_i \nu_i \stackrel{\text{def}}{=} \lambda U. \sum_{i=1}^n r_i \nu_i(U)$  also is in  $\mathcal{V}X$ . The *simple valuations* on  $D$  are those of the form  $\sum_{i=1}^n r_i \delta_{x_i}$ , where  $x_i \in X$ ,  $r_i > 0$ ,  $i = 1, \dots, n$  and  $\sum_{i=1}^n r_i \leq 1$ . The set of simple valuations on  $X$  is denoted by  $\mathcal{S}X$ . Clearly,  $\mathcal{S}X \subseteq \mathcal{V}X$ . Unlike  $\mathcal{V}X$ ,  $\mathcal{S}X$  is not directed-complete in the stochastic order in general.

Given  $\nu \in \mathcal{V}X$  and  $f: X \rightarrow [0, 1]$  continuous, we can define the integral of  $f$  against  $\nu$  by the Choquet formula

$$\int_{x \in X} f(x) d\nu \stackrel{\text{def}}{=} \int_0^1 \nu(f^{-1}((t, 1])) dt,$$

where the right side is a Riemann integral of the bounded antitone function  $\lambda t. \nu(f^{-1}((t, 1]))$ . If no confusion occurs, we simply write  $\int_{x \in X} f(x) d\nu$  as  $\int f d\nu$ . Basic properties of this integral can be found in [11]. Here we note that the map  $\nu \mapsto \int f d\nu: \mathcal{V}X \rightarrow [0, 1]$ , for a fixed  $f$ , is Scott-continuous, and

$$\int f d \sum_i^n r_i \delta_{x_i} = \sum_{i=1}^n r_i f(x_i) \tag{3}$$

for  $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{V}X$ .

For a dcpo  $D$ ,  $\mathcal{V}D$  is defined as  $\mathcal{V}(D, \sigma D)$ . Using Manes' description of monads (Kleisli triples) [21], Jones proved in her PhD thesis [11] that  $\mathcal{V}$  is a monad on **DCPO**:

- The *unit* of  $\mathcal{V}$  at  $D$  is  $\eta_D^\mathcal{V}: D \rightarrow \mathcal{V}D: x \mapsto \delta_x$ .
- The *Kleisli extension*  $f^\dagger$  of a Scott-continuous map  $f: D \rightarrow \mathcal{V}E$  maps  $\nu \in \mathcal{V}D$  to  $f^\dagger(\nu) \in \mathcal{V}E$  by

$$f^\dagger(\nu) \stackrel{\text{def}}{=} \lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu.$$

Then the *multiplication*  $\mu_D^\mathcal{V}: \mathcal{V}\mathcal{V}D \rightarrow \mathcal{V}D$  is given by  $\text{id}_{\mathcal{V}D}^\dagger$ ; it maps  $\varpi \in \mathcal{V}\mathcal{V}D$  to  $\lambda U \in \sigma D. \int_{\nu \in \mathcal{V}D} \nu(U) d\varpi \in \mathcal{V}D$ . Thus,  $\mathcal{V}$  defines an endofunctor on **DCPO** that sends a dcpo  $D$  to  $\mathcal{V}D$ , and a Scott-continuous map  $h: D \rightarrow E$  to  $\mathcal{V}(h) \stackrel{\text{def}}{=} (\eta_E \circ h)^\dagger$ ; concretely,  $\mathcal{V}(h)$  maps  $\nu \in \mathcal{V}D$  to  $\lambda U \in \sigma E. \nu(h^{-1}(U))$ .

Jones [11] also showed that  $\mathcal{V}$  is a strong monad over **DCPO**: its strength at  $(D, E)$  is given by

$$\tau_{DE}^\mathcal{V}: D \times \mathcal{V}E \rightarrow \mathcal{V}(D \times E): (x, \nu) \mapsto \lambda U. \int_{y \in E} \chi_U(x, y) d\nu,$$

where  $\chi_U$  is the characteristic function of  $U \in \sigma(D \times E)$ . Whether  $\mathcal{V}$  is a commutative monad on **DCPO** has remained an open problem for decades. Proving this to be true requires showing the following Fubini-type equation holds:

$$\int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu = \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi, \quad (4)$$

for dcpos  $D$  and  $E$ , for  $U \in \sigma(D \times E)$  and for  $\nu \in \mathcal{V}D, \xi \in \mathcal{V}E$  [22, Section 6]. The difficulty lies in the well-known fact that a Scott open set  $U \in \sigma(D \times E)$  might not be open in the product topology  $\sigma D \times \sigma E$  in general [18, Exercise II-4.26].

However, if either  $\nu$  or  $\xi$  is a simple valuation, then Equation (4) holds. For example, if  $\nu = \sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$ , then by (3) both sides of (4) are equal to  $\sum_{i=1}^n r_i \int_{y \in E} \chi_U(x_i, y) d\xi$ . The Scott continuity of the integral in  $\nu$  then implies Equation (4) holds for valuations that are directed suprema of simple valuations. This is why, for example,  $\mathcal{V}$  is a commutative monad on the category of domains and Scott-continuous maps, as we now explain.

If  $D$  is a dcpo and  $x, y \in D$ , we say  $x$  is *way-below*  $y$  (in symbols,  $x \ll y$ ) if and only if for every directed set  $A$  with  $y \leq \sup A$ , there is some  $a \in A$  such that  $x \leq a$ . We write  $\downarrow y = \{x \in D \mid x \ll y\}$ . A *basis* for a dcpo  $D$  is subset  $B$  satisfying  $\downarrow x \cap B$  is directed and  $x = \sup \downarrow x \cap B$ , for each  $x \in D$ .  $D$  is *continuous* if it has a basis. Continuous dcpos are also called *domains*, and the category of domains and Scott-continuous maps is denoted by **DOM**.

Applying the reasoning above about simple valuations, we obtain a commutative monad of valuations on **DCPO** by restricting to a suitable completion of  $\mathcal{S}D$  inside  $\mathcal{V}D$ . There are several possibilities (cf. [23]), and we choose the smallest and simplest – the d-closure of  $\mathcal{S}D$  in  $\mathcal{V}D$ .

**Definition 4.** For each dcpo  $D$ , we define  $\mathcal{M}D$  to be the intersection of all sub-dcpo's of  $\mathcal{V}D$  that contain  $\mathcal{S}D$ .<sup>1</sup>

Since  $\mathcal{V}D$  itself is a dcpo containing  $\mathcal{S}D$ , it is immediate from the definition of sub-dcpo's that  $\mathcal{M}D$  is a well-defined dcpo in the stochastic order with  $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{V}D$ . Analogous to  $\mathcal{V}D$ ,  $\mathcal{M}D$  also enjoys a convex structure.

**Lemma 5.** For  $\nu_i \in \mathcal{M}D$  and  $r_i \geq 0, i = 1, \dots, n$  with  $\sum_{i=1}^n r_i \leq 1$ , the convex sum  $\sum_{i=1}^n r_i \nu_i$  is still in  $\mathcal{M}D$ .

*Proof.* In Appendix A. □

For the proofs of the following results, we repeatedly use the fact that Scott-continuous maps between dcpos  $D$  and  $E$  are *d-continuous*, i.e., continuous when  $D$  and  $E$  are equipped with the d-topology [24, Lemma 5].

**Theorem 6.**  $\mathcal{M}$  is a commutative monad on **DCPO**.

*Proof.* We sketch the key steps in showing  $\mathcal{M}$  is commutative:

**Unit:** The unit of  $\mathcal{M}$  at  $D$  is  $\eta_D^\mathcal{M}: D \rightarrow \mathcal{M}D: x \mapsto \delta_x$ , the co-restriction of  $\eta_D^\mathcal{V}$  to  $\mathcal{M}D$ . Obviously, it is a well-defined Scott-continuous map.

**Extension:** Since a Scott-continuous map  $f: D \rightarrow \mathcal{M}E$  is also Scott-continuous from  $D$  to  $\mathcal{V}E$ , the Kleisli extension  $f^\dagger: \mathcal{M}D \rightarrow \mathcal{M}E$  of  $f$  can be defined as the restriction and co-restriction of  $f^\dagger: \mathcal{V}D \rightarrow \mathcal{V}E$  to  $\mathcal{M}D$  and  $\mathcal{M}E$ , respectively. The validity of this definition requires  $f^\dagger(\mathcal{M}D) \subseteq \mathcal{M}E$ , which boils down to  $f^\dagger(\mathcal{S}D) \subseteq \mathcal{M}E$  by d-continuity of  $f^\dagger$ , since  $f^\dagger$  is Scott-continuous. Hence we only need to check that  $f^\dagger(\sum_{i=1}^n r_i \delta_{x_i}) \in \mathcal{M}E$  for each  $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$ . However,  $f^\dagger(\sum_{i=1}^n r_i \delta_{x_i}) = \sum_{i=1}^n r_i f(x_i)$ , which is indeed in  $\mathcal{M}E$  by Lemma 5.

<sup>1</sup>The same definition applies in the case of topological spaces.

**Strength:** The strength  $\tau_{DE}^{\mathcal{M}}$  of  $\mathcal{M}$  at  $(D, E)$  is given by  $\tau_{DE}^{\mathcal{V}}$  restricted to  $D \times \mathcal{M}E$  and co-restricted to  $\mathcal{M}(D \times E)$ . This is well-defined provided that  $\tau_{DE}^{\mathcal{V}}$  maps  $D \times \mathcal{M}E$  into  $\mathcal{M}(D \times E)$ . Again, we only need to prove that  $\tau_{DE}^{\mathcal{V}}$  maps  $D \times \mathcal{S}E$  into  $\mathcal{M}(D \times E)$  and conclude the proof with the d-continuity of  $\tau_{DE}^{\mathcal{V}}$  in its second component. Towards this end, we pick  $(a, \sum_{i=1}^n r_i \delta_{y_i}) \in D \times \mathcal{S}E$ , and see

$$\begin{aligned} \tau_{DE}^{\mathcal{V}}(a, \sum_{i=1}^n r_i \delta_{y_i}) &= \lambda U. \int \chi_U(a, y) d \sum_{i=1}^n r_i \delta_{y_i} \\ &\stackrel{(3)}{=} \lambda U. \sum_{i=1}^n r_i \chi_U(a, y_i) \\ &= \lambda U. \sum_{i=1}^n r_i \delta_{(a, y_i)}(U) \stackrel{\text{def}}{=} \sum_{i=1}^n r_i \delta_{(a, y_i)} \end{aligned}$$

is indeed in  $\mathcal{M}(D \times E)$ .

With  $f^\ddagger$  and  $\tau^{\mathcal{M}}$  well-defined, the same arguments used to prove  $(\mathcal{V}, \eta^{\mathcal{V}}, \dashv, \tau^{\mathcal{V}})$  is a strong monad in [11] prove  $(\mathcal{M}, \eta^{\mathcal{M}}, \dashv, \tau^{\mathcal{M}})$  is a strong monad on **DCPO**.

**Commutativity:** Finally, we show  $\mathcal{M}$  is commutative by proving the Equation (4) holds for any dcpo's  $D$  and  $E$  and  $\nu \in \mathcal{M}D, \xi \in \mathcal{M}E$ . As commented above, this holds if  $\nu$  is simple, and then the Scott-continuity of the integral in the  $\nu$ -component implies Equation (4) also holds for directed suprema of simple valuations, directed suprema of directed suprema of simple valuations and so forth, transfinitely. But these are exactly the valuations  $\mathcal{M}D$ .

Formally, we consider for each fixed  $\xi \in \mathcal{M}E$  (even for  $\xi \in \mathcal{V}E$ ) the functions

$$F: \nu \mapsto \int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu: \mathcal{M}D \rightarrow [0, 1]$$

and

$$G: \nu \mapsto \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi: \mathcal{M}D \rightarrow [0, 1].$$

Note that both  $F$  and  $G$  are Scott-continuous functions hence d-continuous, and they are equal on  $\mathcal{S}D$  by Equation (3). Since  $[0, 1]$  is Hausdorff in the d-topology,  $F$  and  $G$  are then equal on the d-closure of  $\mathcal{S}D$  which is, by construction,  $\mathcal{M}D$ .  $\square$

*Remark 7.* The multiplication  $\mu_D^{\mathcal{M}}$  of  $\mathcal{M}$  at  $D$  is given by  $(\text{id}_{\mathcal{M}D})^\ddagger$ . Concretely,  $\mu_D^{\mathcal{M}}$  maps each valuation  $\varpi \in \mathcal{M}(\mathcal{M}D)$  to  $\lambda U \in \sigma D. \int_{\nu \in \mathcal{M}D} \nu(U) d\varpi$ . In particular,  $\mu_D^{\mathcal{M}}$  maps each simple valuation  $\sum_{i=1}^n r_i \delta_{\nu_i} \in \mathcal{M}(\mathcal{M}D)$  to  $\sum_{i=1}^n r_i \nu_i$ , where  $\nu_i \in \mathcal{M}D, i = 1, \dots, n$ , and  $\sum_{i=1}^n r_i \leq 1$ .

*Remark 8.* The double strength of  $\mathcal{M}$  at  $(D, E)$  is given by the Scott-continuous map  $(\nu, \xi) \mapsto \nu \otimes \xi: \mathcal{M}(D) \times \mathcal{M}(E) \rightarrow \mathcal{M}(D \times E)$ , where  $\nu \otimes \xi$  is defined as  $\lambda U \in \sigma(D \times E). \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi$ .

*Remark 9.* We note that  $\mathcal{M}D$  is the first example of a commutative valuations monad on **DCPO** that contains the simple valuations. And, since every valuation on a domain  $D$  is a directed supremum of simple valuations [11, Theorem 5.2], it follows that  $\mathcal{M} = \mathcal{V}$  on the category **DOM**.

### C. Dcpo-completion versus **D**-completion

Recall that a *dcpo-completion* of a poset  $P$  is a pair  $(D, e)$ , where  $D$  is a dcpo and  $e: P \rightarrow D$  is an injective Scott-continuous map, such that for any dcpo  $E$  and Scott-continuous map  $f: P \rightarrow E$ , there exists a unique Scott-continuous map  $f': D \rightarrow E$  satisfying  $f = f' \circ e$ . The dcpo-completion of posets always exists [24, Theorem 1].

As we have seen, for each dcpo  $D$ ,  $\mathcal{M}D$  is the smallest sub-dcpo in  $\mathcal{V}D$  containing  $\mathcal{S}D$ , one may wonder whether  $\mathcal{M}D$ , together with the inclusion map from  $\mathcal{S}D$  into  $\mathcal{M}D$ , is a dcpo-completion of  $\mathcal{S}D$ . The answer is “no” in general. The reason is that the inclusion of  $\mathcal{S}D$  into  $\mathcal{M}D$  may not be Scott-continuous, even when  $D$  is a domain (see [23, Section 6]). The construction  $\mathcal{M}D$  is actually more in a topological flavour, as we now explain. For simplicity, we assume all spaces considered in the sequel are in  $\mathbf{T}_0$ , the category of  $T_0$  spaces and continuous maps.

**Definition 10.** Let  $X$  be a topological space. The *weak topology* on  $\mathcal{V}X$  is generated by the sets

$$[U > r] \stackrel{\text{def}}{=} \{\nu \in \mathcal{V}X \mid \nu(U) > r\},$$

which form a subbasis, where  $U$  is open in  $X$  and  $r \in [0, 1]$ .

*Remark 11.* For each continuous map  $f: X \rightarrow [0, 1]$  and  $r \in [0, 1]$ , the set  $[f > r] \stackrel{\text{def}}{=} \{\nu \in \mathcal{V}X \mid \int f d\nu > r\}$  is open in the weak topology.

We use  $\mathcal{V}_w X$  to denote the space  $\mathcal{V}X$  equipped with the weak topology. We will use the fact that  $\mathcal{V}_w X$  is a sober space, which follows from [25, Proposition 5.1]. It is easy to see that the specialization order on  $\mathcal{V}_w X$  is just the stochastic order. Hence  $\mathcal{V}X = \Omega(\mathcal{V}_w X)$ .

We also use  $\mathcal{S}_w X$  ( $\mathcal{M}_w X$ ) to denote the space  $\mathcal{S}X$  ( $\mathcal{M}X$ ) endowed with the relative topology from  $\mathcal{V}_w X$ . Accordingly,  $\mathcal{M}X = \Omega(\mathcal{M}_w X)$ , and  $\mathcal{S}X = \Omega(\mathcal{S}_w X)$ . Although  $\mathcal{M}X$  is not the dcpo-completion of  $\mathcal{S}X$  in general, we do have the following:

**Proposition 12.** *For each space  $X$ ,  $\mathcal{M}_w X$  is a  $\mathbf{D}$ -completion of  $\mathcal{S}_w X$ . That is,  $\mathcal{M}_w X$  itself is a  $d$ -space, an object in  $\mathbf{D}$ ; the inclusion map  $i: \mathcal{S}_w X \rightarrow \mathcal{M}_w X$  is continuous; and for any  $d$ -space  $Y$  and continuous map  $f: \mathcal{S}_w X \rightarrow Y$ , there exists a unique continuous map  $f': \mathcal{M}_w X \rightarrow Y$  such that  $f = f' \circ i$ .*

The above proposition is a straightforward application of Keimel and Lawson's  $\mathbb{K}$ -category theory [20] to the category  $\mathbf{D}$ .

**Definition 13.** A  $\mathbb{K}$ -category  $\mathbf{K}$  is a full subcategory of  $\mathbf{T}_0$ , whose objects will be called  $k$ -spaces, satisfying:

- 1) Homeomorphic copies of  $k$ -spaces are  $k$ -spaces;
- 2) All sober spaces are  $k$ -spaces, i.e.,  $\mathbf{SOB} \subseteq \mathbf{K}$ ;
- 3) In a sober space  $S$ , the intersection of any family of  $k$ -subspaces, equipped with the relative topology from  $S$ , is a  $k$ -space;
- 4) For any continuous map  $f: S \rightarrow T$  between sober spaces  $S$  and  $T$ , and any  $k$ -subspace  $K$  of  $T$ ,  $f^{-1}(K)$  is  $k$ -subspace of  $S$ .

If  $\mathbf{K}$  is a  $\mathbb{K}$ -category, then the  $\mathbf{K}$ -completion<sup>2</sup> of any  $T_0$ -space  $X$  always exists, and one possible completion process goes as follows [20, Theorem 4.4]: First, pick any  $j: X \rightarrow Y$  such that  $Y$  is sober and  $j$  is a topological embedding. For example, one can take  $j$  as the embedding of  $X$  into its standard sobrification. Second, let  $\tilde{X}$  be the intersection of all  $k$ -subspaces of  $Y$  containing  $j(X)$  and equip it with the relative topology from  $Y$ . Then  $\tilde{X}$ , together with the co-restriction  $i: X \rightarrow \tilde{X}$  of  $j$ , is a  $\mathbf{K}$ -completion of  $X$ .

Now we apply this procedure to prove Proposition 12. First, note that  $\mathbf{D}$  is indeed a  $\mathbb{K}$ -category as proved in [20, Lemma 6.4]. We embed  $\mathcal{S}_w X$  into the sober space  $\mathcal{V}_w X$ , and notice that all  $d$ -subspaces of  $\mathcal{V}_w X$  are precisely sub-dcpo's of  $\mathcal{V}X$ . Hence  $\mathcal{M}_w X$ , which is the intersection of sub-dcpo's  $\mathcal{V}X$  containing  $\mathcal{S}X$  equipped with the relative topology from  $\mathcal{V}_w X$ , is a  $\mathbf{D}$ -completion of  $\mathcal{S}_w X$ .

#### D. A uniform construction

Proposition 12 motivates the next definition.

**Definition 14.** Let  $\mathbf{K}$  be a  $\mathbb{K}$ -category. For each space  $X$ , we define  $\mathcal{V}_{\mathbf{K}} X$  to be the intersection of all  $k$ -subspaces of  $\mathcal{V}_w X$  containing  $\mathcal{S}_w X$ , equipped with the relative topology from  $\mathcal{V}_w X$ .

As discussed above,  $\mathcal{V}_{\mathbf{K}} X$  is a  $\mathbf{K}$ -completion of  $\mathcal{S}_w X$ . It was proved in [23, Theorem 3.5]<sup>3</sup> that  $\mathcal{V}_{\mathbf{K}}: \mathbf{T}_0 \rightarrow \mathbf{T}_0$  is a monad for each  $\mathbb{K}$ -category  $\mathbf{K}$ : The unit of  $\mathcal{V}_{\mathbf{K}}$  at  $X$  maps  $x \in X$  to  $\delta_x$ , and for any continuous map  $f: X \rightarrow \mathcal{V}_{\mathbf{K}} Y$ , the Kleisli extension  $f^\dagger: \mathcal{V}_{\mathbf{K}} X \rightarrow \mathcal{V}_{\mathbf{K}} Y$  maps  $\nu$  to  $\lambda U \in \mathcal{O}Y. \int_{x \in X} f(x)(U) d\nu$ . Therefore, if  $\mathbf{K}$  is a full subcategory of  $\mathbf{D}$ , then according to the construction  $\mathcal{V}_{\mathbf{K}} X$  is always a  $d$ -space for each  $X$ , hence the monad  $\mathcal{V}_{\mathbf{K}}: \mathbf{T}_0 \rightarrow \mathbf{T}_0$  can be restricted to a monad on  $\mathbf{D}$ .

**Theorem 15.** *Let  $\mathbf{K}$  be a  $\mathbb{K}$ -category with  $\mathbf{K} \subseteq \mathbf{D}$ . Then  $\mathcal{V}_{\mathbf{K}, \leq} \stackrel{\text{def}}{=} \Omega \circ \mathcal{V}_{\mathbf{K}} \circ \Sigma$  is a monad on  $\mathbf{DCPO}$ .*

$$\begin{array}{ccc} \mathbf{DCPO} & \xleftrightarrow[\Omega]{\Sigma} & \mathbf{D} & \xleftrightarrow[\mathcal{U}]{\mathcal{F}} & \mathbf{D}^{\mathcal{V}_{\mathbf{K}}} \\ & & \uparrow \mathcal{V}_{\mathbf{K}} & & \\ & & \mathbf{D} & & \end{array}$$

*Proof.* Let  $\mathbf{D}^{\mathcal{V}_{\mathbf{K}}}$  be the Eilenberg-Moore category of  $\mathcal{V}_{\mathbf{K}}$  over  $\mathbf{D}$  and  $\mathcal{F} \dashv \mathcal{U}$  be the adjunction that recovers  $\mathcal{V}_{\mathbf{K}}$ , then  $\mathcal{V}_{\mathbf{K}, \leq} = \Omega \circ \mathcal{U} \circ \mathcal{F} \circ \Sigma$ . The statements follow from the standard categorical fact that adjoints compose:  $\mathcal{F} \circ \Sigma \dashv \Omega \circ \mathcal{U}$ .  $\square$

**Remark 16.** The unit of  $\mathcal{V}_{\mathbf{K}, \leq}$  at dcpo  $D$  sends  $x \in D$  to  $\delta_x$ , and for dcpo's  $D$  and  $E$ , the Kleisli extension  $f^\dagger: \mathcal{V}_{\mathbf{K}, \leq} D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$  of  $f: D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$  maps  $\nu$  to  $\lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu$ .

**Remark 17.**  $\mathcal{M}_w = \mathcal{V}_{\mathbf{D}}$  and  $\mathcal{M} = \mathcal{V}_{\mathbf{D}, \leq}$ .

Note that the category  $\mathbf{SOB}$  of sober spaces is the smallest  $\mathbb{K}$ -category [20, Remark 4.1]. We denote  $\mathcal{V}_{\mathbf{SOB}}$  by  $\mathcal{P}_w$  and  $\mathcal{V}_{\mathbf{SOB}, \leq}$  by  $\mathcal{P}$ . The following statement is then obvious.

<sup>2</sup>The definition of  $\mathbf{K}$ -completion is similar to that of  $\mathbf{D}$ -completion and can be found in [20].

<sup>3</sup>The authors allow valuations to take values in  $[0, \infty]$ . However, the theorem is also true for valuations with values in  $[0, 1]$



**Proposition 18.** Let  $\mathbf{K}$  be a  $\mathbb{K}$ -category with  $\mathbf{K} \subseteq \mathbf{D}$ . Then for each dcpo  $D$ , we have  $SD \subseteq MD \subseteq \mathcal{V}_{\mathbf{K}, \leq} D \subseteq \mathcal{P}D \subseteq \mathcal{V}D$ .

Heckmann [25, Theorem 5.5] proved that  $\mathcal{P}D$  consists of the so-called *point-continuous* valuations on  $D$ . We claim that the Equation 4 holds when either  $\nu$  or  $\xi$  is point-continuous:

**Theorem 19.** Let  $D$  and  $E$  be dcpo's, and  $U \in \sigma(D \times E)$ . Then the equation

$$\int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu = \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi,$$

holds for  $(\nu, \xi) \in \mathcal{P}D \times \mathcal{V}E$  (equivalently,  $(\nu, \xi) \in \mathcal{V}D \times \mathcal{P}E$ ).

As far as we know, this is the most general Fubini theorem on dcpo's. The proof, which relies on the Schröder-Simpson Theorem [26], is included in Appendix A. Hence by combining Remark 16, Proposition 18 and Theorem 19 we get our next theorem.

**Theorem 20.** For any  $\mathbb{K}$ -category  $\mathbf{K}$  with  $\mathbf{K} \subseteq \mathbf{D}$ ,  $\mathcal{V}_{\mathbf{K}, \leq}$  is a commutative monad on **DCPO**.

*Proof.* In Appendix A. □

As promised, we conclude this subsection with a third commutative monad  $\mathcal{W}$  on **DCPO** by describing a  $\mathbb{K}$ -category lying between **SOB** and **D**, the category **WF** consisting of well-filtered spaces and continuous maps. A  $T_0$  space  $X$  is *well-filtered* if, given any filtered family  $\{K_a\}_{a \in A}$  of compact saturated subsets of  $X$  with  $\bigcap_{a \in A} K_a \subseteq U$ , with  $U$  open, there is some  $a \in A$  with  $K_a \subseteq U$ . A proof that **WF** is a  $\mathbb{K}$ -category between **SOB** and **D** can be found in [27]. Hence  $\mathcal{W} \stackrel{\text{def}}{=} \mathcal{V}_{\mathbf{WF}, \leq}$  is a commutative monad on **DCPO** and  $MD \subseteq WD \subseteq \mathcal{P}D$  for every dcpo  $D$ .

*Remark 21.* All subsequent results we present in this paper hold for the three monads  $\mathcal{M}, \mathcal{W}$  and  $\mathcal{P}$ . To avoid cumbersome repetition, we explicitly state them for  $\mathcal{M}$ .

#### E. Continuous Kegelspitzen and $\mathcal{M}$ -algebras

Kegelspitzen [28] are dcpo's that enjoy a convex structure. In this section, we show every *continuous* Kegelspitze  $K$  has a *linear barycenter map*  $\beta: MK \rightarrow K$  making  $(K, \beta)$  an  $\mathcal{M}$ -algebra and conversely, every  $\mathcal{M}$ -algebra  $(K, \beta)$  on **DCPO** admits a Kegelspitze structure on  $K$  making  $\beta: MK \rightarrow K$  a linear map. We begin with the notion of a barycentric algebra.

**Definition 22.** A *barycentric algebra* is a set  $A$  endowed with a binary operation  $a +_r b$  for every real number  $r \in [0, 1]$  such that for all  $a, b, c \in A$  and  $r, p \in [0, 1]$ , the following equations hold:

$$\begin{aligned} a +_1 b &= a; & a +_r b &= b +_{1-r} a; & a +_r a &= a; \\ (a +_p b) +_r c &= a +_{pr} (b +_{\frac{r-p}{1-pr}} c) & & \text{provided } r, p < 1. \end{aligned}$$

**Definition 23.** A *pointed barycentric algebra* is a barycentric algebra  $A$  with a distinguished element  $\perp$ . For  $a \in A$  and  $r \in [0, 1]$ , we define  $r \cdot a \stackrel{\text{def}}{=} a +_r \perp$ . A map  $f: A \rightarrow B$  between pointed barycentric algebras is called *linear* if  $f(\perp_A) = \perp_B$  and  $f(a +_r b) = f(a) +_r f(b)$  for all  $a, b \in A, r \in [0, 1]$ .

**Definition 24.** A *Kegelspitze* is a pointed barycentric algebra  $K$  equipped with a directed-complete partial order such that, for every  $r$  in the unit interval, the functions determined by convex combination  $(a, b) \mapsto a +_r b: K \times K \rightarrow K$  and scalar multiplication  $(r, a) \mapsto r \cdot a: [0, 1] \times K \rightarrow K$  are Scott-continuous in both arguments. A *continuous Kegelspitze* is a Kegelspitze that is a domain in the equipped order.

*Remark 25.* In a Kegelspitze  $K$ , the map  $(r, a) \mapsto r \cdot a = a +_r \perp$  is Scott-continuous, hence monotone, in the  $r$ -component, which implies  $\perp = \perp +_1 a = a +_0 \perp = 0 \cdot a \leq 1 \cdot a = a$  for each  $a \in K$ , i.e.,  $\perp$  is the least element of  $K$ .

**Example 26.** For each dcpo  $D$ ,  $MD$  is a Kegelspitze: for  $\nu_1, \nu_2 \in MD$  and  $r \in [0, 1]$ ,  $\nu_1 +_r \nu_2$  is defined as  $r\nu_1 + (1-r)\nu_2$ . Lemma 5 implies this is well-defined.<sup>4</sup> The constantly zero valuation  $\mathbf{0}_D$  is the distinguished element. Verifying that  $MD$  is a Kegelspitze is then straightforward.

As a consequence, for each Scott-continuous map  $f: D \rightarrow E$ , the map  $\mathcal{M}(f): MD \rightarrow ME: \nu \mapsto \lambda U \in \sigma E. \nu(f^{-1}(U))$  is obviously linear.

**Definition 27.** In each pointed barycentric algebra  $K$ , for  $a_i \in K, r_i \in [0, 1], i = 1, \dots, n$  with  $\sum_{i=1}^n r_i \leq 1$ , we define the convex sum inductively

$$\sum_{i=1}^n r_i a_i \stackrel{\text{def}}{=} \begin{cases} a_1 & , \text{ if } r_1 = 1, \\ a_1 +_{r_1} (\sum_{i=2}^n \frac{r_i}{1-r_1} a_i) & , \text{ if } r_1 < 1. \end{cases}$$

<sup>4</sup>Note that Lemma 5 is stated only for  $\mathcal{M}$ , but it also holds for  $\mathcal{W}$  and  $\mathcal{P}$ : one notes that  $\nu_1 \mapsto r\nu_1 + (1-r)\nu_2: \mathcal{V}_w D \rightarrow \mathcal{V}_w D$  is a continuous map between sober spaces and then uses Definition 13 Item (4) to replace “d-continuity” in the proof.

This is invariant under index-permutation: for  $\pi$  a permutation of  $\{1, \dots, n\}$ ,  $\sum_{i=1}^n r_i a_i = \sum_{i=1}^n r_{\pi(i)} a_{\pi(i)}$  [11, Lemma 5.6]. If  $K$  is a Kegelspitze, then the expression  $\sum_{i=1}^n r_i a_i$  is Scott-continuous in each  $r_i$  and  $a_i$ . A countable convex sum may also be defined: given  $a_i \in K$  and  $r_i \in [0, 1]$ , for  $i \in I$ , with  $\sum_{i \in I} r_i \leq 1$ , let  $\sum_{i \in I} r_i a_i \stackrel{\text{def}}{=} \sup\{\sum_{j \in J} r_j a_j \mid J \subseteq I \text{ and } J \text{ is finite}\}$ .

**Lemma 28.** *A function  $f: K_1 \rightarrow K_2$  between pointed barycentric algebras  $K_1$  and  $K_2$  is linear if and only if  $f(\sum_{i=1}^n r_i a_i) = \sum_{i=1}^n r_i f(a_i)$  for  $a_i \in K_1, i = 1, \dots, n$  and  $\sum_{i=1}^n r_i \leq 1$ .*

**Definition 29.** Let  $K$  be a Kegelspitze and  $s = \sum_{i=1}^n r_i \delta_{x_i}$  be a simple valuation on  $K$ . The *barycenter* of  $s$  is defined as  $\beta_*(s) \stackrel{\text{def}}{=} \sum_{i=1}^n r_i x_i$ .

As a straightforward consequence of Jones' Splitting Lemma ([18, Proposition IV-9.18]), the map  $\beta_*(s)$  is monotone from  $SK$  to  $K$ . If  $K$  is continuous, then  $\mathcal{MK} = \mathcal{VK}$  and  $SK$  is a basis for  $\mathcal{MK}$  (see Remark 9). We extend  $\beta_*$  to the *barycenter map*

$$\beta: \mathcal{MK} \rightarrow K \text{ by } \beta(\nu) \stackrel{\text{def}}{=} \sup\{\beta_*(s) \mid s \in SK \text{ and } s \ll \nu\}.$$

Note that for each simple valuation  $s = \sum_{i=1}^n r_i \delta_{x_i} \in SK$ , there exists a directed set  $A$  of  $SK$  with supremum  $s$  consisting of simple valuations way-below  $s$ . For example, one can choose  $A = \{\sum_{i=1}^n \frac{mr_i}{m+1} \delta_{y_i} \mid m \in \mathbb{N} \text{ and } y_i \ll x_i\}$ . By [18, Lemma IV-9.23.], the map  $\beta$ , as defined above, is a Scott-continuous map extending  $\beta_*$ , i.e.,  $\beta(\nu) = \beta_*(\nu)$  for  $\nu \in SK$ . Moreover,  $\beta$  is a linear map since  $\beta_*$  is.

**Proposition 30.** *Each continuous Kegelspitze  $K$  admits a linear barycenter map  $\beta: \mathcal{MK} \rightarrow K$  (as above) for which the pair  $(K, \beta)$  is an Eilenberg-Moore algebra of  $\mathcal{M}$ .*

*Proof.* Clearly,  $\beta \circ \eta_K^{\mathcal{M}} = \text{id}_K$ . To prove that  $\beta \circ \mu_K^{\mathcal{M}} = \beta \circ \mathcal{M}(\beta)$ , we only need to prove both sides are equal on simple valuations in  $\mathcal{M}(\mathcal{MK})$ , since  $\mathcal{S}(\mathcal{MK})$  is dense in  $\mathcal{M}(\mathcal{MK})$  in the d-topology, and both sides of the equation are d-continuous functions. However, when applied to the simple valuation  $\sum_{i=1}^n r_i \delta_{\nu_i} \in \mathcal{S}(\mathcal{MK})$ , both sides equal  $\sum_{i=1}^n r_i \beta(\nu_i)$ . This follows from direct computation by employing Remark 7 and linearity of  $\beta$ .  $\square$

We next show that every Eilenberg-Moore algebra  $(K, \beta)$  of  $\mathcal{M}$  on **DCPO** admits a Kegelspitze structure on  $K$  making  $\beta: \mathcal{MK} \rightarrow K$  a linear map.

**Proposition 31.** *Let  $(K, \beta)$  be an  $\mathcal{M}$ -algebra on **DCPO**. For  $a, b \in K$  and  $r \in [0, 1]$ , define  $a +_r b \stackrel{\text{def}}{=} \beta(\delta_a +_r \delta_b)$ . Then with the operation  $+_r$ ,  $K$  is a Kegelspitze and  $\beta: \mathcal{MK} \rightarrow K$  is linear.*

*Proof.* See Appendix A.  $\square$

**Proposition 32.** *Let  $(K_1, \beta_1)$  and  $(K_2, \beta_2)$  be  $\mathcal{M}$ -algebras on **DOM**. A Scott-continuous function  $f: K_1 \rightarrow K_2$  is an algebra morphism from  $(K_1, \beta_1)$  to  $(K_2, \beta_2)$  if and only if  $f$  is linear with respect to the Kegelspitze structure on  $K_1$  and  $K_2$  introduced by  $\beta_1$  and  $\beta_2$ , respectively, as in Proposition 31.*

*Proof.* See Appendix A.  $\square$

**Theorem 33.** *The Eilenberg-Moore category  $\mathbf{DOM}^{\mathcal{M}}$  of  $\mathcal{M}$  over **DOM** is isomorphic to the category of continuous Kegelspitzen and Scott-continuous linear maps.*

*Proof.* Combine Propositions 30, 31 and 32.  $\square$

*Remark 34.* Theorem 33 characterises  $\mathbf{DOM}^{\mathcal{M}}$ , which equals  $\mathbf{DOM}^{\mathcal{V}_{\mathbf{K}, \leq}}$  for any  $\mathbf{K}$ -category  $\mathbf{K}$  with  $\mathbf{K} \subseteq \mathbf{D}$  since  $\mathcal{V} = \mathcal{M}$  on domains (see Remark 9 and Proposition 18). This corrects an error in [11]: there it is proved that *continuous abstract probabilistic domains* and linear maps form a full subcategory of  $\mathbf{DOM}^{\mathcal{V}}$ . But there is a claim that all objects in  $\mathbf{DOM}^{\mathcal{V}}$  are abstract probabilistic domains. A separating example is the extended non-negative reals  $[0, \infty]$ , which is a continuous Kegelspitze but not an abstract probabilistic domain.

## F. The Kleisli Category

In this subsection we describe the categorical properties of the Kleisli category of our monad  $\mathcal{M}$ . Everything we say in this subsection is also true for our other two monads as well.

We write  $\mathbf{DCPO}_{\mathcal{M}}$  for the Kleisli category of our monad  $\mathcal{M}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}$ . In order to distinguish between the categorical primitives of **DCPO** and  $\mathbf{DCPO}_{\mathcal{M}}$ , we indicate with  $f: A \rightarrow B$  the morphisms of  $\mathbf{DCPO}_{\mathcal{M}}$  and we write  $f \circ g \stackrel{\text{def}}{=} \mu \circ \mathcal{M}(f) \circ g$  for the Kleisli composition of morphisms in  $\mathbf{DCPO}_{\mathcal{M}}$ . We write  $\text{id}_A: A \rightarrow A$  with  $\text{id}_A = \eta_A: A \rightarrow \mathcal{M}A$  for the identity morphisms in  $\mathbf{DCPO}_{\mathcal{M}}$ . The monad  $\mathcal{M}$  induces an adjunction  $\mathcal{J} \dashv \mathcal{U}: \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$ , where:

$$\mathcal{J}A \stackrel{\text{def}}{=} A, \quad \mathcal{J}f \stackrel{\text{def}}{=} \eta \circ f, \quad \mathcal{U}A \stackrel{\text{def}}{=} \mathcal{M}A, \quad \mathcal{U}f \stackrel{\text{def}}{=} \mu \circ \mathcal{M}f.$$

1) *Coproducts*: The category  $\mathbf{DCPO}_{\mathcal{M}}$  inherits (small) coproducts from  $\mathbf{DCPO}$  in the standard way [29, pp. 264] and we write  $A_1 \dot{+} A_2 \stackrel{\text{def}}{=} A_1 + A_2$  for the induced (binary) coproduct. The induced coprojections are given by  $\mathcal{J}(in_1): A_1 \rightarrow A_1 \dot{+} A_2$  and  $\mathcal{J}(in_2): A_2 \rightarrow A_1 \dot{+} A_2$ . Then for  $f: A \rightarrow C$  and  $g: B \rightarrow D$ ,  $f \dot{+} g = [\mathcal{M}(in_C) \circ f, \mathcal{M}(in_D) \circ g]$ .

2) *Symmetric monoidal structure*: Because our monad  $\mathcal{M}$  is *commutative*, it induces a symmetric monoidal structure on  $\mathbf{DCPO}_{\mathcal{M}}$  in a canonical way [30, pp. 462]. The induced tensor product is  $A \dot{\times} B \stackrel{\text{def}}{=} A \times B$  and the Kleisli projections are  $\mathcal{J}(\pi_A): A \dot{\times} B \rightarrow A$  and  $\mathcal{J}(\pi_B): A \dot{\times} B \rightarrow B$ <sup>5</sup>. For  $f: A \rightarrow C$  and  $g: B \rightarrow D$ , their tensor product is given by  $f \dot{\times} g = \lambda(a, b).f(a) \otimes g(b)$ . Note that the last expression uses the double strength of  $\mathcal{M}$ , see Remark 8.

Standard categorical arguments now show that the Kleisli products distribute over the Kleisli coproducts. We write  $d_{A,B,C}: A \dot{\times} (B \dot{+} C) \cong (A \dot{\times} B) \dot{+} (A \dot{\times} C)$  for this natural isomorphism.

3) *The left adjoint  $\mathcal{J}$* : The functor  $\mathcal{J}$ , whose action is the identity on objects, preserves the monoidal structure and the coproduct structure up to equality (and not merely up to isomorphism). That is,  $\mathcal{J}(A \star B) = JA \star JB$  and  $\mathcal{J}(f \star g) = Jf \star Jg$ , where  $\star \in \{\times, +\}$ .

4) *Kleisli Exponential*: Our Kleisli adjunction also contains the structure of a *Kleisli-exponential* (which is also known as a  *$\mathcal{M}$ -exponential*). Following Moggi [14], we will use this to interpret higher-order function types. Next, we describe this structure in greater detail.

The functor  $J(-) \dot{\times} B: \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  has a right adjoint, which we write as  $[B \rightarrow -]: \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$ , for each dcpo  $B$ . In particular  $[B \rightarrow -] \stackrel{\text{def}}{=} [B \rightarrow \mathcal{U}(-)]$ , which means that, on objects,  $[B \rightarrow C] = [B \rightarrow \mathcal{M}C]$ . This data provides us with a family of Scott-continuous bijections

$$\lambda: \mathbf{DCPO}_{\mathcal{M}}(\mathcal{J}A \dot{\times} B, C) \cong \mathbf{DCPO}(A, [B \rightarrow C]) \quad (5)$$

natural in  $A$  and  $C$ , called *currying*. We also denote with  $\epsilon: \mathcal{J}[B \rightarrow -] \dot{\times} B \Rightarrow \text{Id}$ , the counit of the adjunctions (5), often called *evaluation*. Because this family of adjunctions is parameterised by objects  $B$  of  $\mathbf{DCPO}_{\mathcal{M}}$ , it follows using standard categorical results [31, §IV.7] that the assignment  $[B \rightarrow -]: \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$  may be extended uniquely to a bifunctor  $[- \rightarrow -]: \mathbf{DCPO}_{\mathcal{M}}^{\text{op}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$ , such that the bijections  $\lambda$  in (5) are natural in all components<sup>6</sup>.

*Remark 35.* Some authors describe currying and evaluation for Kleisli exponentials without referring to the functor  $\mathcal{J}$ . This cannot lead to confusion on the object level, but to be fully precise, one has to specify that the naturality properties on the  $A$ -component hold only for *total maps*. We make this explicit by including  $\mathcal{J}$  in our presentation.

5) *Enrichment Structure*: The category  $\mathbf{DCPO}_{\mathcal{M}}$  is enriched over  $\mathbf{DCPO}_{\perp 1}$ : for all dcpo's  $A, B$  and  $C$ , the Kleisli exponential  $[A \rightarrow B] = [A \rightarrow \mathcal{M}B] = \mathbf{DCPO}_{\mathcal{M}}(A, B)$  is a pointed dcpo in the pointwise order, and the Kleisli composition

$$\circ: [A \rightarrow B] \times [B \rightarrow C] \rightarrow [A \rightarrow C]: (f, g) \mapsto g \circ f = g^{\dagger} \circ f$$

is obviously a strict Scott-continuous map. Moreover, the adjunction  $\mathcal{J} \dashv \mathcal{U}: \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$  is also  $\mathbf{DCPO}$ -enriched (see [32, Definition 6.7.1] for definition) and so are the bifunctors  $(- \dot{\times} -), (- \dot{+} -)$  and  $[- \rightarrow -]$ .

We interpret probabilistic effects using the convex structure of our model which we now describe. For each dcpo  $B$ ,  $\mathcal{M}B$  is a Kegelspitze in the stochastic order (Example 26): for  $r \in [0, 1]$  and  $\nu_1, \nu_2 \in \mathcal{M}B$ ,  $\nu_1 +_r \nu_2$  is defined as  $r\nu_1 + (1-r)\nu_2$ ; the zero-valuation  $\mathbf{0}_B$  is the distinguished element (which is also least). It follows that  $[A \rightarrow B] = \mathbf{DCPO}_{\mathcal{M}}(A, B)$  is a Kegelspitze in the pointwise order: for  $f, g \in [A \rightarrow B]$ ,  $f +_r g$  is defined as  $\lambda x.f(x) +_r g(x)$ . Next, we note that this convex structure is preserved by Kleisli composition  $\circ$ , Kleisli coproduct  $\dot{+}$  and Kleisli product  $\dot{\times}$ .

**Lemma 36.** *Let  $A, B, C, D$  be dcpo's,  $f, f_1, f_2 \in [A \rightarrow B]$ ,  $g, g_1, g_2 \in [B \rightarrow C]$ ,  $h, h_1, h_2 \in [C \rightarrow D]$  and  $r \in [0, 1]$ . Then we have:*

- $(g_1 +_r g_2) \circ f = g_1 \circ f +_r g_2 \circ f$ ;
- $g \circ (f_1 +_r f_2) = g \circ f_1 +_r g \circ f_2$ ;
- $(f_1 +_r f_2) \dot{\star} h = f_1 \dot{\star} h +_r f_2 \dot{\star} h$ ;
- $f \dot{\star} (h_1 +_r h_2) = f \dot{\star} h_1 +_r f \dot{\star} h_2$ ,

where  $\star \in \{\times, +\}$  in the last two cases.

*Proof.* See Appendix C. □

6) *Important Subcategories*: In order to describe our denotational semantics, we have to identify two important subcategories of  $\mathbf{DCPO}_{\mathcal{M}}$ .

<sup>5</sup>These projections do *not* satisfy the universal property of a product.

<sup>6</sup>This extension is canonically given by  $[f \rightarrow g] \stackrel{\text{def}}{=} \lambda(g \circ \epsilon_{C_1} \circ (\mathbf{id} \dot{\times} f))$ .

**Definition 37.** The subcategory of *deterministic total maps*, denoted **TD**, is the full-on-objects subcategory of  $\mathbf{DCPO}_{\mathcal{M}}$  each of whose morphisms  $f: X \rightarrow Y$  admits a factorisation  $f = \mathcal{J}(f') = \left( X \xrightarrow{f'} Y \xrightarrow{\eta_Y} \mathcal{M}Y \right)$ .

Therefore, by definition, each map  $f: X \rightarrow Y$  in **TD** satisfies  $f(x) = \delta_y$  for some  $y \in Y$ . These maps are *deterministic* in the sense that they carry no interesting convex structure and they are *total* in the sense that they map all inputs  $x \in X$  to non-zero valuations. The importance of this subcategory is that all values of our language admit an interpretation within **TD**. Moreover, the categorical structure of **TD** is very easy to describe, as our next proposition shows.

**Proposition 38.** *There exists a DCPO-enriched isomorphism of categories  $\mathbf{DCPO} \cong \mathbf{TD}$ .*

*Proof.* Each map  $\eta_X: X \rightarrow \mathcal{M}X$  is injective, because  $\Sigma X$  is a  $T_0$  space and so  $\mathcal{J}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  is faithful. Its corestriction to **TD** is the required isomorphism.  $\square$

In our model, the canonical copy map at an object  $A$  is given by the map  $\mathcal{J}\langle \text{id}_A, \text{id}_A \rangle: A \rightarrow A \times A$  and the canonical discarding map at  $A$  is the map  $\mathcal{J}(1_A): A \rightarrow 1$ , where  $1_A: A \rightarrow 1$  is the terminal map of **DCPO**. Because maps in **TD** are in the image of  $\mathcal{J}$ , it follows that they are compatible with the copy and discard maps and thus also with weakening and contraction [5], [6].

The next subcategory we introduce is important, because we will use it for the interpretation of open types. It has sufficient structure to solve recursive domain equations.

**Definition 39.** The subcategory of *deterministic partial maps*, denoted **PD**, is the full-on-objects subcategory of  $\mathbf{DCPO}_{\mathcal{M}}$  each of whose morphisms  $f: X \rightarrow Y$  admits a factorisation  $f = \left( X \xrightarrow{f'} Y_{\perp} \xrightarrow{\phi_Y} \mathcal{M}Y \right)$ , where  $Y_{\perp}$  is the dcpo obtained from  $Y$  by freely adding a least element  $\perp$ , and  $\phi_Y$  is the map:

$$\phi_Y: Y_{\perp} \rightarrow \mathcal{M}Y :: y \mapsto \begin{cases} \mathbf{0}_Y & , \text{ if } y = \perp \\ \delta_y & , \text{ if } y \neq \perp . \end{cases}$$

These maps are *partial* because some inputs are mapped to  $\mathbf{0}$ , but also deterministic, because the convex structure is trivial in both cases. This is further justified by the next proposition.

**Proposition 40.** *There exists a DCPO $_{\perp}$ -enriched isomorphism of categories  $\mathbf{DCPO}_{\mathcal{L}} \cong \mathbf{PD}$ , where  $\mathbf{DCPO}_{\mathcal{L}}$  is the Kleisli category of the lift monad  $\mathcal{L}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}$ .*

*Proof.* The assignment  $\phi$  from Definition 39 is a *strong map of monads*  $\phi: \mathcal{L} \Rightarrow \mathcal{M}$  which then induces a functor  $\mathcal{F}: \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  (Appendix B). Each  $\phi_Y$  is injective, so the corestriction of  $\mathcal{F}$  to **PD** is the required isomorphism.  $\square$

7) *Solving Recursive Domain Equations:* In order to interpret recursive types, we solve the required recursive domain equations by constructing *parameterised initial algebras* [9], [10] within (the subcategory of embeddings of) **PD** using the limit-colimit coincidence theorem [33].

**Definition 41** (see [9, §6.1]). Given a category  $\mathbf{C}$  and a functor  $\mathcal{T}: \mathbf{C}^{n+1} \rightarrow \mathbf{C}$ , a *parameterised initial algebra* for  $\mathcal{T}$  is a pair  $(\mathcal{T}^{\sharp}, \iota^{\mathcal{T}})$ , such that:

- $\mathcal{T}^{\sharp}: \mathbf{C}^n \rightarrow \mathbf{C}$  is a functor;
- $\iota^{\mathcal{T}}: \mathcal{T} \circ \langle \text{Id}, \mathcal{T}^{\sharp} \rangle \Rightarrow \mathcal{T}^{\sharp}: \mathbf{C}^n \rightarrow \mathbf{C}$  is a natural transformation;
- For every  $\vec{C} \in \text{Ob}(\mathbf{C}^n)$ , the pair  $(\mathcal{T}^{\sharp}\vec{C}, \iota^{\mathcal{T}}_{\vec{C}})$  is an initial  $\mathcal{T}(\vec{C}, -)$ -algebra.

In the special case when  $n = 1$ , we recover the usual notion of initial algebra. We consider *parameterised* initial algebras because we need to interpret mutual type recursion. Similarly, one can also define the dual notion of *parameterised final coalgebra*.

**Proposition 42** (see [16, §4.3]). *Let  $\mathbf{C}$  be a category with an initial object and all  $\omega$ -colimits and let  $\mathcal{T}: \mathbf{C}^{n+1} \rightarrow \mathbf{C}$  be an  $\omega$ -cocontinuous functor. Then  $\mathcal{T}$  has a parameterised initial algebra  $(\mathcal{T}^{\sharp}, \iota^{\mathcal{T}})$  and the functor  $\mathcal{T}^{\sharp}: \mathbf{C}^n \rightarrow \mathbf{C}$  is also  $\omega$ -cocontinuous.*

The next proposition shows that the subcategory **PD** has sufficient structure to solve recursive domain equations.

**Proposition 43.** *The subcategory **PD** is (parameterised) DCPO-algebraically compact. More specifically, every DCPO-enriched functor  $\mathcal{T}: \mathbf{PD}^{n+1} \rightarrow \mathbf{PD}$  has a parameterised compact algebra, i.e., a parameterised initial algebra whose inverse is a parameterised final coalgebra for  $\mathcal{T}$ .*

*Proof.* By Proposition 40, we have  $\mathbf{PD} \cong \mathbf{DCPO}_{\mathcal{L}} \cong \mathbf{DCPO}_{\perp,1}$  and the latter two categories are well-known to be **DCPO**-algebraically compact (which may be easily established using [9, Corollary 7.2.4]).  $\square$

Therefore, every **DCPO**-enriched *covariant* functor on  $\mathbf{DCPO}_{\mathcal{M}}$  which restricts to  $\mathbf{PD}$  can be equipped with a parameterised compact algebra. In order to solve equations involving *mixed-variance* functors (induced by function types), we use the limit-colimit coincidence theorem [33]. In particular, an important observation made by Smyth and Plotkin in [33] allows us to interpret all type expressions (including function spaces) as covariant functors on *subcategories of embeddings*. These ideas are developed in detail in [15], [16] and here we also follow this approach.

**Definition 44.** Given a **DCPO**-enriched category  $\mathbf{C}$ , an *embedding* of  $\mathbf{C}$  is a morphism  $e: X \rightarrow Y$ , such that there exists (a necessarily unique) morphism  $e^p: Y \rightarrow X$ , called a *projection*, with the properties:  $e^p \circ e = \text{id}_X$  and  $e \circ e^p \leq \text{id}_Y$ . We denote with  $\mathbf{C}_e$  the full-on-objects subcategory of  $\mathbf{C}$  whose morphisms are the embeddings of  $\mathbf{C}$ .

**Proposition 45.** *The category  $\mathbf{PD}_e$  has an initial object and all  $\omega$ -colimits, and the following assignments:*

- $\dot{\times}_e: \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$  by  
 $X \dot{\times}_e Y \stackrel{\text{def}}{=} X \dot{\times} Y$  and  $e_1 \dot{\times}_e e_2 \stackrel{\text{def}}{=} e_1 \dot{\times} e_2$ .
- $\dot{+}_e: \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$  by  
 $X \dot{+}_e Y \stackrel{\text{def}}{=} X \dot{+} Y$  and  $e_1 \dot{+}_e e_2 \stackrel{\text{def}}{=} e_1 \dot{+} e_2$
- $[\rightarrow]_e^{\mathcal{J}}: \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$   
 $[X \rightarrow Y]_e^{\mathcal{J}} \stackrel{\text{def}}{=} \mathcal{J}[X \rightarrow Y]$  and  $[e_1 \rightarrow e_2]_e^{\mathcal{J}} \stackrel{\text{def}}{=} \mathcal{J}[e_1^p \rightarrow e_2]$

define covariant  $\omega$ -cocontinuous bifunctors on  $\mathbf{PD}_e$ .

*Proof.* This follows using results from [33] together with some restriction arguments which we present in Appendix B.  $\square$

Therefore, by Proposition 42 and Proposition 45 we can solve recursive domain equations induced by all well-formed type expressions (with no restrictions on the admissible logical polarities of the types) within  $\mathbf{PD}_e$ . However, since our judgements support weakening and contraction, we have an extra proof obligation: showing each isomorphism that is a solution to a recursive domain equation can be copied and discarded. This is indeed true (for any isomorphism in  $\mathbf{PD}$ ) because of the next proposition.

**Proposition 46.** *Every isomorphism of  $\mathbf{PD}$  (and  $\mathbf{PD}_e$ ) is also an isomorphism of  $\mathbf{TD}$ .*

*Proof.* In Appendix B.  $\square$

We have already explained that morphisms of  $\mathbf{TD}$  are compatible with weakening and contraction, so the above proposition suffices for our purposes.

#### IV. INDUCTIVE TYPES AND QUANTUM EFFECTS

An *operator algebra* is a Banach algebra of bounded linear operators on a Hilbert space. Operator algebras are a standard framework for describing quantum mechanics, within which the commutative algebras describe classical systems. Thus they offer the ideal setting to study the interaction between the classical and quantum worlds. This is important for quantum computing, where one simultaneously deals with classical data that can be both copied or discarded, and quantum data that is not copyable. The most important classes of operator algebras are  $C^*$ -algebras and von Neumann algebras, which can be regarded as dual to ‘non-commutative’ topological spaces and ‘non-commutative’ measure spaces, respectively. This is in analogy to the categories of commutative  $C^*$ -algebras and of commutative von Neumann algebras, which are dual to the categories of locally compact Hausdorff spaces and of measure spaces, respectively. For quantum computing, it is sufficient to consider those von Neumann algebras that can be regarded as the non-commutative generalization of sets: the hereditarily atomic von Neumann algebras, i.e., (possibly infinite) products of matrix algebras. These algebras were studied by Kornell [34], who also showed they form a category whose dual has a concrete description: quantum sets. We refer to [35], [36] and [37] as standard references on operator algebras.

##### A. Definition of von Neumann algebras

Let  $H$  be a Hilbert space. Any linear map  $x: H \rightarrow H$  is called an *operator*, which is called *bounded* if it is continuous with respect to the norm  $\|\cdot\|$  induced by the inner product  $\langle \cdot, \cdot \rangle$  on  $H$ . The space of all bounded operators on  $H$  is denoted by  $B(H)$ , and forms an algebra over  $\mathbb{C}$  if we define multiplication by composition. Moreover,  $B(H)$  is equipped with an unary operation  $x \mapsto x^*$ , the *involution*, where  $x^*$  is the unique bounded operator such that  $\langle x^*k, h \rangle = \langle k, xh \rangle$  for each  $h, k \in H$ .

Any subalgebra  $A$  of  $B(H)$  closed under the involution is called a *\*-subalgebra*. If  $S \subseteq B(H)$  is a subset, then we define its *commutant*  $S' = \{y \in B(H) \mid xy = yx \ (\forall x \in S)\}$ .

**Definition 47.** Let  $H$  be a Hilbert space. A *\*-subalgebra*  $M$  of  $B(H)$  such that  $M'' = M$  is called a *von Neumann algebra* on  $H$ . If  $K$  is another Hilbert space, and  $N$  is a von Neumann algebra on  $K$ , we call a linear map  $\varphi : M \rightarrow N$  preserving the multiplication and the involution a *\*-homomorphism*. If, in addition,  $\varphi$  is bijective, we call it a *\*-isomorphism*.

Since the commutant of any non-empty set in  $B(H)$  always contains  $1_H$ , it follows that  $1_H \in M$ . If we want to emphasize that it is the unit element of  $M$ , we write  $1_M$  instead of  $1_H$ .

**Example 48.**  $B(H)$  itself is a von Neumann algebra, and if  $H$  is  $n$ -dimensional, then  $B(H)$  is *\*-isomorphic* to the  $n \times n$ -complex valued matrices. This example plays an important role in the definition of hereditarily atomic von Neumann algebras below.

**Example 49.** Let  $X$  be a set. Then  $\ell^2(X)$ , the space of all functions  $f : X \rightarrow \mathbb{C}$  such that  $\sum_{x \in X} |f(x)|^2 < \infty$  is a Hilbert space with inner product  $\langle f, g \rangle \stackrel{\text{def}}{=} \sum_{x \in X} \overline{f(x)}g(x)$ . The space  $\ell^\infty(X)$  of all functions  $f : X \rightarrow \mathbb{C}$  such that  $\sup_{x \in X} |f(x)| < \infty$  equipped with the norm  $\|f\| \stackrel{\text{def}}{=} \sup_{x \in X} |f(x)|$  can be embedded isometrically into  $B(\ell^2(X))$  via the maps  $f \mapsto m_f$ , where  $m_f : \ell^2(X) \rightarrow \ell^2(X)$  is the left multiplication  $g \mapsto fg$  [38, Proposition B.73]. If we identify  $\ell^\infty(X)$  with its image in  $B(\ell^2(X))$ , it is a commutative von Neumann algebra on  $\ell^2(X)$  [38, Proposition B.108].

Given a collection  $(H_\alpha)_{\alpha \in A}$  of Hilbert spaces, the sum  $\bigoplus_{\alpha \in A} H_\alpha = \{(h_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} H_\alpha \mid \sum_{\alpha \in A} \|h_\alpha\|^2 < \infty\}$  is a Hilbert space whose inner product is given by  $\langle k, h \rangle \stackrel{\text{def}}{=} \sum_{\alpha \in A} \langle k_\alpha, h_\alpha \rangle$  for  $k = (k_\alpha)_{\alpha \in A}$  and  $h = (h_\alpha)_{\alpha \in A}$ .

**Proposition 50.** [36, Proposition II.3.3] Let  $(M_\alpha)_{\alpha \in A}$  be a collection of von Neumann algebras where for each  $\alpha \in A$ ,  $M_\alpha$  is a von Neumann algebra on some Hilbert space  $H_\alpha$ . Then  $\prod_{\alpha \in A} M_\alpha \stackrel{\text{def}}{=} \{(x_\alpha)_{\alpha \in A} \mid \sup_{\alpha \in A} \|x_\alpha\| < \infty\}$  is a von Neumann algebra on  $H \stackrel{\text{def}}{=} \bigoplus_{\alpha \in A} H_\alpha$  where  $xh \stackrel{\text{def}}{=} (x_\alpha h_\alpha)_{\alpha \in A} \in H$  for  $x = (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} M_\alpha$  and  $h = (h_\alpha)_{\alpha \in A} \in H$ .

**Definition 51.** We call a von Neumann algebra  $M$  *hereditarily atomic* if  $M$  is isomorphic to  $\prod_{\alpha \in A} M_\alpha$ , where each  $M_\alpha$  is *\*-isomorphic* to some matrix algebra.

Let  $X$  be a set, then it follows from the definition of  $\ell^\infty(X)$  that it is isomorphic to  $\prod_{x \in X} \mathbb{C}$ , whence  $\ell^\infty(X)$  is hereditarily atomic.

## B. Categorical Structure of Hereditarily Atomic von Neumann algebras

In this subsection we describe the structure of the category of hereditarily atomic von Neumann algebras we use in our semantics. To define the appropriate morphisms, we first need some extra structure. Let  $M$  be a von Neumann algebra on a Hilbert space  $H$ . Then we say that  $x \in M$  is *self adjoint* if  $x^* = x$ , and *positive* if  $x = y^*y$  for some  $y \in M$ , or equivalently if  $\langle h, xh \rangle \geq 0$  for each  $h \in H$  [37, Theorem 4.2.6]. Given self-adjoint elements  $x$  and  $y$  in  $M$ , we write  $x \leq y$  if  $y - x$  is positive. The relation  $\leq$  is a partial order on the set  $M_{\text{sa}}$  of self-adjoint elements in  $M$  under which  $M_{\text{sa}}$  is *bounded directed complete*: any bounded monotonically ascending net  $(x_\alpha)_{\alpha \in A}$  in  $M_{\text{sa}}$  (i.e.,  $(\exists y \in M_{\text{sa}}) x_\alpha \leq y \ (\forall \alpha \in A)$ ) has a supremum  $\sup_{\alpha \in A} x_\alpha = x \in M_{\text{sa}}$ . Then  $x$  is the limit of  $(x_\alpha)$  in the *strong operator topology* on  $M$ : i.e.,  $\lim_{\alpha \in A} x_\alpha h = xh$  for each  $h \in H$ ; this also implies convergence with respect to the *weak operator topology* on  $M$ , i.e.,  $\lim_{\alpha \in A} \langle k, x_\alpha h \rangle = \langle k, xh \rangle$  for each  $h, k \in H$  [35, Proposition I.3.2.5 & Corollary I.3.2.6]. As a consequence, the *unit interval* of  $M$ ,  $[0, 1]_M = \{x \in M_{\text{sa}} \mid 0 \leq x \leq 1_M\}$  is a dcpo. A linear function  $\varphi : M \rightarrow N$  between von Neumann algebras is *unital* if  $\varphi(1_M) = 1_N$ , *subunital* if  $\varphi(1_M) \leq 1_N$ , and *positive* if it preserves positive elements, or equivalently, if it is monotone with respect to  $\leq$ . If  $\varphi$  is positive and subunital, it restricts to a monotone map  $[0, 1]_M \rightarrow [0, 1]_N$ , which by linearity completely determines  $\varphi$ . We call  $\varphi$  *normal* if it preserves the suprema of bounded increasing nets, i.e., if it is Scott continuous with respect to  $\leq$ .

We denote by  $M_n(M)$  the von Neumann algebra of all  $n \times n$ -matrices with entries in  $M$ . Any linear map  $\varphi : M \rightarrow N$  between von Neumann algebras induces a linear map  $\varphi^{(n)} : M_n(M) \rightarrow M_n(N)$  obtained by applying  $\varphi$  entrywise. We say that  $\varphi$  is *completely positive* if  $\varphi^{(n)}$  is positive for each  $n \in \mathbb{N}$ . In particular, any completely positive map is positive.

Von Neumann algebras have an intrinsic topology which is known under several names: the  *$\sigma$ -weak operator topology*, the *ultraweak operator topology* or the *weak\*-topology*. Any completely positive map between von Neumann algebras is normal if and only if it is continuous with respect to the  $\sigma$ -weak operator topology [35, Proposition III.2.2.2]. The  $\sigma$ -weak operator topology is stronger than the weak operator topology, but both topologies coincide on norm-bounded subsets [35, I.3.1.4].

We note that the bicommutant theorem of von Neumann states that a *\*-subalgebra* of  $B(H)$  is a von Neumann algebra if and only if it is closed with respect to either the weak operator topology, the strong operator topology or the  $\sigma$ -weak operator topology (and hence with respect to all of them).

We now introduce the following categories:

**Definition 52.** We denote the category of von Neumann algebras and normal completely positive subunital maps by  $\mathbf{vN}$ . We denote its wide subcategory of von Neumann algebras and normal  $*$ -homomorphisms by  $\mathbf{vN}_*$  and its full subcategory of hereditarily atomic von Neumann algebras and normal completely positive subunital maps by  $\mathbf{HA}$ . We denote the wide subcategory of  $\mathbf{HA}$  consisting of all hereditarily atomic von Neumann algebras with normal  $*$ -homomorphisms by  $\mathbf{HA}_*$ . We denote the formal duals of  $\mathbf{HA}$  and  $\mathbf{HA}_*$  by  $\mathbf{Q}$  and  $\mathbf{Q}_*$ , respectively.

Rather than  $\mathbf{HA}$  and  $\mathbf{HA}_*$ , the categories  $\mathbf{Q}$  and  $\mathbf{Q}_*$  are relevant for our semantics, for the following reasons. Recall the program of non-commutative geometry is based on dualities involving categories of operator algebras, for instance Gelfand duality between the category of commutative unital  $C^*$ -algebras and  $*$ -homomorphisms, and the category of compact Hausdorff spaces and continuous functions. As a result, instead of the category of unital  $C^*$ -algebras itself, its formal dual, the category of ‘non-commutative’ compact Hausdorff spaces, is the appropriate category. Since  $\mathbf{Set}$  is dual to the category of commutative hereditarily atomic von Neumann algebras and normal  $*$ -homomorphisms,  $\mathbf{Q}_*$ , the dual of  $\mathbf{HA}_*$  is the appropriate non-commutative generalization of  $\mathbf{Set}$ <sup>7</sup>. While  $\mathbf{HA}_*$  has all products, it is monoidal co-closed, i.e., the monoidal product has a left instead of a right adjoint. Since we require a monoidal closed category with coproducts, we are forced to use  $\mathbf{Q}_*$  instead of  $\mathbf{Q}$ . We now describe the categorical structures on  $\mathbf{Q}$  and  $\mathbf{Q}_*$ .

1) *Coproducts:* Proposition 50 describes the categorical product on  $\mathbf{vN}_*$ , which restricts to a categorical product on  $\mathbf{HA}_*$  since the product of hereditarily atomic von Neumann algebras clearly is hereditarily atomic. Given  $M_1, M_2, N \in \mathbf{HA}$  and normal  $*$ -homomorphisms  $\varphi : N \rightarrow M_1$  and  $\psi : N \rightarrow M_2$ , the product map  $\langle \varphi, \psi \rangle : N \rightarrow M_1 \times M_2$  is given by  $\langle \varphi, \psi \rangle(x) = (\varphi(x), \psi(x))$  for each  $x \in N$ . It is easy to see that if  $\varphi$  and  $\psi$  are normal completely positive subunital maps, then  $\langle \varphi, \psi \rangle(x) = (\varphi(x), \psi(x))$  defines a normal completely positive subunital map, whence the product on  $\mathbf{HA}_*$  extends to a product on  $\mathbf{HA}$ . As a consequence,  $\mathbf{Q}$  and  $\mathbf{Q}_*$  have binary coproducts.

2) *Symmetric Monoidal Structure:* Given two von Neumann algebras  $M$  and  $N$  on Hilbert spaces  $H$  and  $K$ , respectively, the algebraic tensor product  $M \odot N$  acts in a natural way on the Hilbert space tensor product  $H \otimes K$ . The weak operator closure of  $M \odot N$  in  $B(H \otimes K)$  is a von Neumann algebra, which we denote by  $M \bar{\otimes} N$ , and which we call the *spatial tensor product* of  $M$  and  $N$ . The construction in [35, III.2.2.5] shows the spatial tensor product of von Neumann algebras induces a symmetric monoidal product on both  $\mathbf{HA}_*$  and  $\mathbf{HA}$ , hence on  $\mathbf{Q}_*$  and  $\mathbf{Q}$ .

**Theorem 53.** [34, Theorem 9.1]  $\mathbf{Q}_*$  is symmetric monoidal closed.

3) *The adjunction:* The embedding  $\mathcal{I} : \mathbf{Q}_* \rightarrow \mathbf{Q}$  corresponds to an embedding  $\mathbf{HA}_* \rightarrow \mathbf{HA}$  that is shown to have a left adjoint in [39, Section 4.3.4], so  $\mathcal{I}$  has a right adjoint. We note that a similar statement for all von Neumann algebras is shown by the same author in [40, Example 14]. We denote by  $\mathcal{T}$  the induced monad on  $\mathbf{Q}_*$ . A functor  $\mathcal{F}$  is said to be *Kleislian* if it has a right adjoint, and the comparison functor from the induced Kleisli category into the codomain of  $\mathcal{F}$  is an isomorphism. It is shown in [40, Theorem 9] that  $\mathcal{F}$  is Kleislian if and only if it is bijective on objects and it has a right adjoint. Thus  $\mathcal{I}$  is clearly Kleislian, whence  $\mathbf{Q}$  is isomorphic to the Kleisli category of the monad  $\mathcal{T}$  on  $\mathbf{Q}_*$ . Since the monoidal product on  $\mathbf{Q}$  restricts to the monoidal product on  $\mathbf{Q}_*$ , it follows that  $\mathcal{I}$  is a strict monoidal functor. Since  $\mathcal{I}$  is a left adjoint, it preserves the coproduct up to isomorphism.

4) *(Parameterised) initial algebras:* We know show that the category  $\mathbf{Q}_*$  has sufficient structure for the construction of parameterised initial algebras.

**Proposition 54.** The category  $\mathbf{Q}_*$  is cocomplete and the functors  $\otimes_* : \mathbf{Q}_* \times \mathbf{Q}_* \rightarrow \mathbf{Q}_*$  and  $\oplus_* : \mathbf{Q}_* \times \mathbf{Q}_* \rightarrow \mathbf{Q}_*$  are cocontinuous.

*Proof.* Cocompleteness of  $\mathbf{Q}_*$  is shown in [34, Propositions 8.6], whence the coproduct bifunctor  $\oplus_*$  is cocontinuous. Cocontinuity of  $\otimes_*$  is due to the fact that it has a right adjoint as follows from Theorem 53.  $\square$

5) *Enrichment structure:* The category  $\mathbf{vN}$  is enriched over  $\mathbf{DCPO}_{\perp!}$  [41], if we define an order on the each homset  $\mathbf{vN}(M, N)$  by  $\varphi \leq \psi$  if and only if  $\psi - \varphi$  is completely positive for each  $\varphi, \psi \in \mathbf{vN}(M, N)$  between von Neumann algebras  $M$  and  $N$ . We note that by [42, Proposition 4.2] we have  $\varphi \leq \psi$  if and only if  $\psi - \varphi$  is positive for each  $\varphi, \psi \in \mathbf{vN}(M, N)$ . The  $\mathbf{DCPO}_{\perp!}$ -enrichment of  $\mathbf{vN}$  immediately implies the  $\mathbf{DCPO}_{\perp!}$ -enrichment of  $\mathbf{HA}$  and  $\mathbf{Q}$ . We claim that the latter categories are actually enriched over continuous dcpos. A proof is given in Appendix D.

**Theorem 55.**  $\mathbf{Q}$  is enriched over continuous dcpo’s.

**Theorem 56.** The category  $\mathbf{Q}$  is enriched over continuous *Kegelspitzen*.

## V. DENOTATIONAL SEMANTICS

We now give the denotational semantics of our language.

<sup>7</sup>We note that [34] introduces a concrete category  $\mathbf{qSet}$  of so-called *quantum sets* and functions, which is shown to be dual to  $\mathbf{HA}_*$  [34, Theorem 7.6], hence equivalent to  $\mathbf{Q}_*$ .

$$\begin{aligned}
\llbracket \Theta \vdash \mathbf{A} \rrbracket &: \mathbf{Q}_*^{|\Theta|} \rightarrow \mathbf{Q}_* & \llbracket \Theta \vdash \Theta_i \rrbracket &\stackrel{\text{def}}{=} \Pi_i & \llbracket \Theta \vdash \mathbf{I} \rrbracket &\stackrel{\text{def}}{=} K_C & \llbracket \Theta \vdash \mathbf{qbit} \rrbracket &\stackrel{\text{def}}{=} K_{M_2(\mathbb{C})} & \llbracket \Theta \vdash \mu\mathbf{X}.\mathbf{A} \rrbracket &\stackrel{\text{def}}{=} \llbracket \Theta, \mathbf{X} \vdash \mathbf{A} \rrbracket^\# \\
\llbracket \Theta \vdash \mathbf{A} \oplus \mathbf{B} \rrbracket &\stackrel{\text{def}}{=} \oplus_* \circ \langle \llbracket \Theta \vdash \mathbf{A} \rrbracket, \llbracket \Theta \vdash \mathbf{B} \rrbracket \rangle & \llbracket \Theta \vdash \mathbf{A} \otimes \mathbf{B} \rrbracket &\stackrel{\text{def}}{=} \otimes_* \circ \langle \llbracket \Theta \vdash \mathbf{A} \rrbracket, \llbracket \Theta \vdash \mathbf{B} \rrbracket \rangle \\
\llbracket \Theta \vdash P \rrbracket &: \mathbf{PD}_e^{|\Theta|} \rightarrow \mathbf{PD}_e & \llbracket \Theta \vdash \Theta_i \rrbracket &\stackrel{\text{def}}{=} \Pi_i & \llbracket \Theta \vdash 1 \rrbracket &\stackrel{\text{def}}{=} K_1 & \llbracket \Theta \vdash \mu X.P \rrbracket &\stackrel{\text{def}}{=} \llbracket \Theta, X \vdash P \rrbracket^\# \\
\llbracket \Theta \vdash P + R \rrbracket &\stackrel{\text{def}}{=} \dot{+}_e \circ \langle \llbracket \Theta \vdash P \rrbracket, \llbracket \Theta \vdash R \rrbracket \rangle & \llbracket \Theta \vdash P \times R \rrbracket &\stackrel{\text{def}}{=} \dot{\times}_e \circ \langle \llbracket \Theta \vdash P \rrbracket, \llbracket \Theta \vdash R \rrbracket \rangle \\
\llbracket \Theta \vdash P \rightarrow R \rrbracket &\stackrel{\text{def}}{=} [\rightarrow]_e^{\mathcal{J}} \circ \langle \llbracket \Theta \vdash P \rrbracket, \llbracket \Theta \vdash R \rrbracket \rangle & \llbracket \Theta \vdash Q(\mathbf{A}, \mathbf{B}) \rrbracket &\stackrel{\text{def}}{=} K_{\mathbf{Q}(\llbracket \mathbf{A} \rrbracket, \llbracket \mathbf{B} \rrbracket)}
\end{aligned}$$

Fig. 11. Interpretation of types.  $K_X$  is the constant- $X$ -functor. Given closed types  $\mathbf{A}$  and  $P$ , we write  $\llbracket \mathbf{A} \rrbracket \stackrel{\text{def}}{=} [\cdot \vdash \mathbf{A}](*)$  and  $\llbracket P \rrbracket \stackrel{\text{def}}{=} [\cdot \vdash P](*)$ .

$$\begin{aligned}
\llbracket \mathbf{I} \rrbracket &= \mathbb{C} & \llbracket \mathbf{A} \otimes \mathbf{B} \rrbracket &= \llbracket \mathbf{A} \rrbracket \otimes \llbracket \mathbf{B} \rrbracket & \llbracket \mathbf{A} \oplus \mathbf{B} \rrbracket &= \llbracket \mathbf{A} \rrbracket \oplus \llbracket \mathbf{B} \rrbracket & \llbracket \mathbf{qbit} \rrbracket &= M_2(\mathbb{C}) \\
\llbracket 1 \rrbracket &= 1 & \llbracket P \times R \rrbracket &= \llbracket P \rrbracket \times \llbracket R \rrbracket & \llbracket P + R \rrbracket &= \llbracket P \rrbracket + \llbracket R \rrbracket & \llbracket Q(\mathbf{A}, \mathbf{B}) \rrbracket &= \mathbf{Q}(\llbracket \mathbf{A} \rrbracket, \llbracket \mathbf{B} \rrbracket) \\
\llbracket P \rightarrow R \rrbracket &= \llbracket \llbracket P \rrbracket \rightarrow \mathcal{M}[\llbracket R \rrbracket] \rrbracket
\end{aligned}$$

Fig. 12. Derived equations for closed types.

$$\begin{aligned}
\llbracket \Phi \vdash m : P \rrbracket &: \llbracket \Phi \rrbracket \rightarrow \llbracket P \rrbracket \text{ in } \mathbf{DCPO}_{\mathcal{M}} \\
\llbracket \Phi, x : P \vdash x : P \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\pi_2 \\
\llbracket \Phi \vdash () : 1 \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}1 \\
\llbracket \Phi \vdash (m, n) : P \times R \rrbracket &\stackrel{\text{def}}{=} \text{dst} \circ \langle \llbracket m \rrbracket, \llbracket n \rrbracket \rangle \\
\llbracket \Phi \vdash \pi_i m : P_i \rrbracket &\stackrel{\text{def}}{=} \mathcal{M}\pi_i \circ \llbracket m \rrbracket, \text{ for } i \in \{1, 2\} \\
\llbracket \Phi \vdash in_i m : P_1 + P_2 \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}in_i \circ \llbracket m \rrbracket, \text{ for } i \in \{1, 2\} \\
\llbracket \Phi \vdash (\text{case } m \text{ of } in_1 x \Rightarrow n_1 \mid in_2 y \Rightarrow n_2) : R \rrbracket &\stackrel{\text{def}}{=} \llbracket \llbracket n_1 \rrbracket, \llbracket n_2 \rrbracket \rrbracket \circ \cong \circ (\text{id} \dot{\times} \llbracket m \rrbracket) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\
\llbracket \Phi \vdash \lambda x^P. m : P \rightarrow R \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\lambda(\llbracket m \rrbracket) \\
\llbracket \Phi \vdash mn : R \rrbracket &\stackrel{\text{def}}{=} \epsilon \circ (\llbracket m \rrbracket \dot{\times} \llbracket n \rrbracket) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\
\llbracket \Phi \vdash \mathbf{qfun}(\mathbf{x}_1, \dots, \mathbf{x}_n) \Rightarrow \mathbf{q} : Q(\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_n, \mathbf{B}) \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\llbracket \mathbf{q} \rrbracket \\
\llbracket \Phi \vdash \text{new} : Q(\mathbf{bit}, \mathbf{qbit}) \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}(x \mapsto \text{new}) \\
\llbracket \Phi \vdash U : Q(\mathbf{qbit}^{\otimes n}, \mathbf{qbit}^{\otimes n}) \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}(x \mapsto \text{unitary}_U) \\
\llbracket \Phi \vdash \text{meas} : Q(\mathbf{qbit}, \mathbf{bit}) \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}(x \mapsto \text{meas}) \\
\llbracket \Phi \vdash \text{fold } m : \mu X.P \rrbracket &\stackrel{\text{def}}{=} \text{fold} \circ \llbracket m \rrbracket \\
\llbracket \Phi \vdash \text{unfold } m : P[\mu X.P/X] \rrbracket &\stackrel{\text{def}}{=} \text{unfold} \circ \llbracket m \rrbracket \\
\llbracket \Phi \vdash \text{run } \mathcal{C} : \mathbf{O} \rrbracket &\stackrel{\text{def}}{=} r_{\mathbf{O}} \circ \llbracket \mathcal{C} \rrbracket
\end{aligned}$$

Fig. 13. Interpretation of classical term judgements.

### A. Interpretation of Types

We begin with the interpretation of (open) types, which is described in Figure 11. The following result follows by induction using Propositions 42, 45 and 54.

**Proposition 57.** *The assignments  $\llbracket \Theta \vdash \mathbf{A} \rrbracket : \mathbf{Q}_*^{|\Theta|} \rightarrow \mathbf{Q}_*$  and  $\llbracket \Theta \vdash P \rrbracket : \mathbf{PD}_e^{|\Theta|} \rightarrow \mathbf{PD}_e$  are  $\omega$ -cocontinuous functors.*

As a special case, for closed types, we see that  $\llbracket \mathbf{A} \rrbracket \in \text{Ob}(\mathbf{Q}_*) = \text{Ob}(\mathbf{Q})$  and  $\llbracket P \rrbracket \in \text{Ob}(\mathbf{PD}_e) = \text{Ob}(\mathbf{DCPO})$  and the equations in Figure 12 follow immediately.

We proceed by defining the folding/unfolding isomorphisms for recursive types and proving a necessary lemma.

**Lemma 58** (Substitution). *Given quantum types  $\Theta, \mathbf{X} \vdash \mathbf{A}$  and  $\Theta \vdash \mathbf{B}$  and classical types  $\Theta, X \vdash P$  and  $\Theta \vdash R$ , then:*

$$\begin{aligned}
\llbracket \Theta \vdash \mathbf{A}[\mathbf{B}/\mathbf{X}] \rrbracket &= \llbracket \Theta, \mathbf{X} \vdash \mathbf{A} \rrbracket \circ \langle \text{Id}, \llbracket \Theta \vdash \mathbf{B} \rrbracket \rangle & \text{and} \\
\llbracket \Theta \vdash P[R/X] \rrbracket &= \llbracket \Theta, X \vdash P \rrbracket \circ \langle \text{Id}, \llbracket \Theta \vdash R \rrbracket \rangle.
\end{aligned}$$

**Definition 59.** For closed types  $\mu\mathbf{X}.\mathbf{A}$  and  $\mu X.P$ , we define:

$$\begin{aligned}
\text{fold}_{\mu\mathbf{X}.\mathbf{A}} &: \llbracket \mathbf{A}[\mu\mathbf{X}.\mathbf{A}/\mathbf{X}] \rrbracket = \llbracket \mathbf{X} \vdash \mathbf{A} \rrbracket \llbracket \mu\mathbf{X}.\mathbf{A} \rrbracket \cong \llbracket \mu\mathbf{X}.\mathbf{A} \rrbracket \\
\text{fold}_{\mu X.P} &: \llbracket P[\mu X.P/X] \rrbracket = \llbracket X \vdash P \rrbracket \llbracket \mu X.P \rrbracket \cong \llbracket \mu X.P \rrbracket,
\end{aligned}$$

where the equalities are from Lemma 58 and the isomorphisms are the initial algebra structures. We write  $\text{unfold}_{\mu\mathbf{X}.\mathbf{A}}$  and  $\text{unfold}_{\mu X.P}$  for the inverse isomorphisms. Note that  $\text{fold}_{\mu X.P}$  and  $\text{fold}_{\mu\mathbf{X}.\mathbf{A}}$  are isomorphisms in  $\mathbf{TD}$  and  $\mathbf{Q}_*$ , respectively.



$$\begin{aligned}
& \llbracket \Phi; \Gamma \vdash \mathbf{q} : \mathbf{A} \rrbracket : \llbracket \Phi \rrbracket \rightarrow \mathbf{Q}(\llbracket \Gamma \rrbracket, \llbracket \mathbf{A} \rrbracket) \text{ in } \mathbf{DCPO} \\
& \llbracket \Phi; \mathbf{y} : \mathbf{A} \vdash \mathbf{y} : \mathbf{A} \rrbracket \stackrel{\text{def}}{=} x \mapsto \text{id}_{\llbracket \mathbf{A} \rrbracket} \\
& \llbracket \Phi; \cdot \vdash * : \mathbf{I} \rrbracket \stackrel{\text{def}}{=} x \mapsto \text{id}_{\mathbb{C}} \\
& \llbracket \Phi; \Gamma_1, \Gamma_2 \vdash \mathbf{q}; \mathbf{r} : \mathbf{A} \rrbracket \stackrel{\text{def}}{=} x \mapsto (\cong \circ (\llbracket \mathbf{q} \rrbracket_x \otimes \llbracket \mathbf{r} \rrbracket_x)) \\
& \llbracket \Phi; \Gamma_1, \Gamma_2 \vdash \mathbf{q} \otimes \mathbf{r} : \mathbf{A} \oplus \mathbf{B} \rrbracket \stackrel{\text{def}}{=} x \mapsto \llbracket \mathbf{q} \rrbracket_x \otimes \llbracket \mathbf{r} \rrbracket_x \\
& \llbracket \Phi; \Gamma_1, \Gamma_2 \vdash \text{let } \mathbf{x} \otimes \mathbf{y} = \mathbf{q} \text{ in } \mathbf{r} : \mathbf{B} \rrbracket \stackrel{\text{def}}{=} x \mapsto \llbracket \mathbf{r} \rrbracket_x \circ (\text{id}_{\llbracket \Gamma_2 \rrbracket} \otimes \llbracket \mathbf{q} \rrbracket_x) \circ \text{swap} \\
& \llbracket \Phi; \Gamma \vdash \text{in}_i \mathbf{q} : \mathbf{A} \oplus \mathbf{B} \rrbracket \stackrel{\text{def}}{=} x \mapsto \text{in}_i \circ \llbracket \mathbf{q} \rrbracket_x \\
& \llbracket \Phi; \Gamma_1, \Gamma_2 \vdash (\text{case } \mathbf{q} \text{ of } \text{in}_1 \mathbf{x} \Rightarrow \mathbf{r}_1 \mid \text{in}_2 \mathbf{y} \Rightarrow \mathbf{r}_2) : \mathbf{B} \rrbracket \stackrel{\text{def}}{=} x \mapsto \llbracket \mathbf{r}_1 \rrbracket_x, \llbracket \mathbf{r}_2 \rrbracket_x \circ \cong \circ (\text{id} \otimes \llbracket \mathbf{q} \rrbracket_x) \circ \text{swap} \\
& \llbracket \Phi; \Gamma \vdash m\mathbf{q} : \mathbf{B} \rrbracket \stackrel{\text{def}}{=} x \mapsto \beta(\llbracket m \rrbracket_x) \circ \llbracket \mathbf{q} \rrbracket_x \\
& \llbracket \Phi; \Gamma \vdash \text{fold } \mathbf{q} : \mu \mathbf{X}. \mathbf{A} \rrbracket \stackrel{\text{def}}{=} x \mapsto \text{fold} \circ \llbracket \mathbf{q} \rrbracket_x \\
& \llbracket \Phi; \Gamma \vdash \text{unfold } \mathbf{q} : \mathbf{A}[\mu \mathbf{X}. \mathbf{A} / \mathbf{X}] \rrbracket \stackrel{\text{def}}{=} x \mapsto \text{unfold} \circ \llbracket \mathbf{q} \rrbracket_x \\
& \llbracket \Phi; \cdot \vdash \text{init } m : \mathbf{O} \rrbracket \stackrel{\text{def}}{=} r_{\mathbf{O}}^{-1} \circ \llbracket m \rrbracket \\
& \llbracket \Phi; \Gamma_1, \Gamma_2 \vdash \text{let } x = \text{lift } \mathbf{q} \text{ in } \mathbf{r} : \mathbf{A} \rrbracket \stackrel{\text{def}}{=} x \mapsto \widehat{\llbracket \mathbf{r} \rrbracket}(x) \circ (\llbracket \mathbf{q} \rrbracket_x \otimes \text{id}_{\llbracket \Gamma_2 \rrbracket})
\end{aligned}$$

Fig. 14. Interpretation of quantum term judgements.

### B. Relationship between Quantum and Classical Probabilistic Effects

Our language shows that quantum effects induce probabilistic effects on the classical side (via the "run" term) and, vice-versa, probabilistic effects on the classical side can also influence the quantum dynamics (via the "init" and "mq" terms). In this section, we describe the mathematical structures we use to interpret this correspondence.

By Proposition 30 and Theorem 56, we can define a map  $\beta_{\mathbf{A}, \mathbf{B}} : \mathcal{MQ}(\llbracket \mathbf{A} \rrbracket, \llbracket \mathbf{B} \rrbracket) \rightarrow \mathbf{Q}(\llbracket \mathbf{A} \rrbracket, \llbracket \mathbf{B} \rrbracket)$ , for all quantum types  $\mathbf{A}$  and  $\mathbf{B}$ . This is used for the interpretation of the "mq" term. Intuitively, the barycenter maps  $\beta$  allow us to combine probabilistic and quantum effects into quantum effects.

The interpretations of the "run" and "init" terms rely on an important isomorphism, which we describe next.

**Proposition 60.** *Let  $\mathbf{O}$  be an observable and closed quantum type. Then, there exists an isomorphism of Kegelspitzen  $r_{\mathbf{O}} : \mathbf{Q}(\mathbb{C}, \llbracket \mathbf{O} \rrbracket) \cong \mathcal{M}(\llbracket \mathbf{O} \rrbracket) : r_{\mathbf{O}}^{-1}$ .*

*Proof.* We present a sketch of the proof, full details are in Appendix E. First, one can show that for every classical observable type  $\llbracket \mathbf{O} \rrbracket$ , we have that  $\llbracket \llbracket \mathbf{O} \rrbracket \rrbracket$  is a *discrete* dcpo. Writing  $\llbracket \llbracket \mathbf{O} \rrbracket \rrbracket$  for its underlying set, then one can show  $\llbracket \mathbf{O} \rrbracket \cong \ell^\infty(\llbracket \llbracket \mathbf{O} \rrbracket \rrbracket)$ , which is a *commutative* von Neumann algebra. Standard results from operator algebras show that, for any set  $X$ ,  $\mathbf{Q}(\mathbb{C}, \ell^\infty(X)) \cong \mathcal{DX}$ , where  $\mathcal{DX}$  is the Kegelspitzen of all discrete subprobability distributions on  $X$ . The proof is concluded after recognising that  $\mathcal{DX} = \mathcal{M}(X, \sqsubseteq)$ , where  $\sqsubseteq$  is the discrete order on  $X$ .  $\square$

Intuitively, this isomorphism shows that there is a 1-1 correspondence between the quantum and probabilistic states of *observable* types. Finally, we construct an isomorphism which we use for the interpretation of the "lift" term. This is similar to a construction first reported in [43].

**Proposition 61.** *Given a classical context  $\Phi$ , quantum context  $\Gamma$ , quantum type  $\mathbf{A}$  and observable quantum type  $\mathbf{O}$ , there exists a Scott-continuous bijection*

$$\widehat{(-)} : \mathbf{DCPO}(\llbracket \Phi \rrbracket \times \llbracket \llbracket \mathbf{O} \rrbracket \rrbracket, \mathbf{Q}(\llbracket \Gamma \rrbracket, \llbracket \mathbf{A} \rrbracket)) \cong \mathbf{DCPO}(\llbracket \Phi \rrbracket, \mathbf{Q}(\llbracket \mathbf{O} \rrbracket \otimes \llbracket \Gamma \rrbracket, \llbracket \mathbf{A} \rrbracket)).$$

*Proof.* In Appendix F.  $\square$

### C. Interpretation of Terms and Quantum Configurations

A classical context  $\Phi = x_1 : P_1, \dots, x_n : P_n$  is interpreted as the dcpo  $\llbracket \Phi \rrbracket \stackrel{\text{def}}{=} \llbracket P_1 \rrbracket \times \dots \times \llbracket P_n \rrbracket$ . A quantum context  $\Gamma = \mathbf{x}_1 : \mathbf{A}_1, \dots, \mathbf{x}_n : \mathbf{A}_n$  is interpreted as the HA algebra  $\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \llbracket \mathbf{A}_1 \rrbracket \otimes \dots \otimes \llbracket \mathbf{A}_n \rrbracket$ . The interpretation of classical/quantum term judgements and quantum configurations is defined by mutual induction in Figures 13 and 14 and Equation 6. Next, we explain some of the notation used therein.

The interpretation of a classical term judgement  $\Phi \vdash m : P$  is a morphism  $\llbracket \Phi \vdash m : P \rrbracket : \llbracket \Phi \rrbracket \rightarrow \llbracket P \rrbracket$  in  $\mathbf{DCPO}_{\mathcal{M}}$  and we shall often abbreviate this by writing  $\llbracket m \rrbracket$ .

A quantum term judgement  $\Phi; \Gamma \vdash \mathbf{q} : \mathbf{A}$  is interpreted as a morphism  $\llbracket \Phi; \Gamma \vdash \mathbf{q} : \mathbf{A} \rrbracket : \llbracket \Phi \rrbracket \rightarrow \mathbf{Q}(\llbracket \Gamma \rrbracket, \llbracket \mathbf{A} \rrbracket)$  in  $\mathbf{DCPO}$ . We often abbreviate this by writing  $\llbracket \mathbf{q} \rrbracket$ . For an element  $x \in \llbracket \Phi \rrbracket$ , we also write  $\llbracket \mathbf{q} \rrbracket_x$  and  $\llbracket m \rrbracket_x$  as a shorthand for  $\llbracket \mathbf{q} \rrbracket(x)$  and  $\llbracket m \rrbracket(x)$ , respectively. In the special case when  $\Phi = \cdot$ , then we will see  $\llbracket \mathbf{q} \rrbracket$  as a morphism  $\llbracket \mathbf{q} \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \mathbf{A} \rrbracket$  in  $\mathbf{Q}$ .

The interpretation of a configuration  $\Phi \vdash [\psi, \ell, \mathbf{q}] : \mathbf{A}$  is a morphism  $[[\Phi \vdash [\psi, \ell, \mathbf{q}] : \mathbf{A}]] : [[\Phi]] \rightarrow \mathbf{Q}(\mathbf{C}, [[\mathbf{A}]] \otimes \mathbf{qbit}^{\otimes k})$  in **DCPO**, where  $k = \dim(\psi) - |\ell|$ , defined by:

$$[[\Phi \vdash [\psi, \ell, \mathbf{q}] : \mathbf{A}]] \stackrel{\text{def}}{=} x \mapsto ([[q]]_x \otimes \text{id}_{\mathbf{qbit}^{\otimes k}}) \circ \text{state}_{\psi, \psi^*} \quad (6)$$

In the special case when  $\mathcal{C} = [\psi, \ell, \mathbf{q}]$  is total, then its interpretation can be seen as a morphism  $[[\mathcal{C}]] : [[\Phi]] \rightarrow \mathbf{Q}(\mathbf{C}, [[\mathbf{A}]])$  in **DCPO**, defined by  $(x \mapsto ([[q]]_x \circ \text{state}_{\psi, \psi^*}))$ . If, moreover,  $\Phi = \cdot$ , then we can see it as a map  $[[\mathcal{C}]] : \mathbf{C} \rightarrow [[\mathbf{A}]]$  in  $\mathbf{Q}$ .

#### D. Soundness and Computational Adequacy

We can now state the main semantic results for our model. The interpretations of both classical terms and quantum configurations are Scott-continuous functions whose codomain is a Kegelspitze and so convex sums over them may be defined pointwise in the obvious way.

The interpretation of values in our language enjoys additional structural properties.

**Lemma 62.** *For any classical value  $\Phi \vdash v : P$  and quantum value  $\Phi; \Gamma \vdash \mathbf{v} : \mathbf{A}$ , we have:*

- 1)  $[[v]] : [[\Phi]] \rightarrow [[P]]$  is also a morphism of **TD**. Equivalently, it is in the image of  $\mathcal{J}$ .
- 2)  $[[\mathbf{v}]] : [[\Phi]] \rightarrow \mathbf{Q}([[ \Gamma ]], [[ \mathbf{A} ]])$  corestricts to  $\mathbf{Q}_*([[ \Gamma ]], [[ \mathbf{A} ]])$ . That is,  $\forall x \in [[\Phi]]. [[\mathbf{v}]](x) \in \mathbf{Q}_*([[ \Gamma ]], [[ \mathbf{A} ]])$ .

As usual, before we can prove soundness, we have to prove a substitution lemma.

**Lemma 63** (Substitution). *Let  $\Phi \vdash v : P$  be a classical value and  $\Phi; \Sigma \vdash \mathbf{v} : \mathbf{A}$  a quantum value. Then:*

- 1) For any classical term  $\Phi, x : P \vdash m : R$ , we have  $[[m[v/x]]] = [[m]] \circ (\text{id}_{[[\Phi]]} \times [[v]]) \circ \mathcal{J}(\text{id}_{[[\Phi]]}, \text{id}_{[[\Phi]]})$ .
- 2) For any quantum term  $\Phi; \Gamma, \mathbf{y} : \mathbf{A} \vdash \mathbf{q} : \mathbf{B}$  and any  $x \in [[\Phi]]$ , we have  $[[\mathbf{q}[\mathbf{v}/\mathbf{y}]]](x) = [[\mathbf{q}]](x) \circ (\text{id}_{[[\Gamma]]} \otimes [[\mathbf{v}]](x))$ .
- 3) For any quantum term  $\Phi, z : P; \Gamma \vdash \mathbf{q} : \mathbf{B}$  and any  $x \in [[\Phi]]$ , we have  $[[\mathbf{q}[v/z]]](x) = [[\mathbf{q}]](x, [[v]](x))$ .

Soundness is the statement that our interpretation is invariant under single-step reduction (in a probabilistic sense).

**Theorem 64** (Soundness). *For any classical term  $\Phi \vdash m : P$  and quantum configuration  $\Phi \vdash \mathcal{C} : \mathbf{A}$ , we have:*

$$[[m]] = \sum_{m \xrightarrow{p} m'} p[[m']] \quad [[\mathcal{C}]] = \sum_{\mathcal{C} \xrightarrow{r} \mathcal{C}'} r[[\mathcal{C}']]$$

assuming  $m \xrightarrow{p} m'$  and  $\mathcal{C} \xrightarrow{r} \mathcal{C}'$  for some rules from the operational semantics (Section II) and where the convex sums range over all such rules.

In the above theorem, both sums have at most two summands. The next, considerably stronger statement, generalises this result for reductions involving an arbitrary number of steps, and so the convex sums may have an infinite support.

**Theorem 65** (Strong Adequacy). *Let  $\cdot \vdash m : P$  be a classical term and  $\cdot \vdash \mathcal{C} : \mathbf{A}$  a total quantum configuration. Then:*

$$[[m]] = \sum_{v \in \text{Val}(P)} P(m \rightarrow_* v) [[v]]$$

$$[[\mathcal{C}]] = \sum_{\mathcal{V} \in \text{ValC}(\mathbf{A})} P(\mathcal{C} \rightarrow_* \mathcal{V}) [[\mathcal{V}]] \quad (\text{see Definition 27}),$$

where  $\text{Val}(P) \stackrel{\text{def}}{=} \{w \mid \cdot \vdash w : P \text{ is a value}\}$  and  $\text{ValC}(\mathbf{A}) \stackrel{\text{def}}{=} \{\mathcal{W} \mid \cdot \vdash \mathcal{W} : \mathbf{A} \text{ is a total value configuration}\}$ .

*Proof.* In Appendix G. **TODO: finish the quantum part of the proof** □

**Corollary 66** (Adequacy). *Let  $\cdot \vdash m : 1$  be a classical term and  $\cdot \vdash \mathbf{q} : \mathbf{I}$  a quantum term, both of unit types. Then:*

$$[[m]](1) = \text{Halt}(m) \quad \text{and} \quad [[\mathbf{q}]]_1(1) = \text{Halt}(\mathbf{q}),$$

where the right-hand sides  $\text{Halt}(m) \stackrel{\text{def}}{=} P(m \rightarrow_* ())$  and  $\text{Halt}(\mathbf{q}) \stackrel{\text{def}}{=} P([1, \emptyset, \mathbf{q}] \rightarrow_* [1, \emptyset, *])$  are, in fact, the probabilities that programs  $m$  and  $\mathbf{q}$  terminate, respectively.

## VI. CONCLUSION

We described a mixed linear/non-linear quantum/probabilistic programming language which is suitable for programming hybrid quantum-classical algorithms. We provided a formal syntax and operational semantics for the language which is type-safe. We also described a sound and adequate denotational semantics. In order to do so, we interpret the classical probabilistic effects using a commutative probabilistic monad on **dcpo**'s. Quantum resources and effects are interpreted in the category of hereditarily atomic von Neumann algebras which we proved is enriched over continuous domains. The relationship between quantum and classical probabilistic effects is modelled via novel semantic methods which use the barycentre maps of our model that are well-behaved under the strong sense of enrichment we established for the quantum semantics.

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APPENDIX A  
MONADS, COMMUTATIVITY AND  $\mathcal{M}$ -ALGEBRAS

Let  $D$  be a dcpo. Recall that the  $d$ -topology on  $D$  consists of all sub-dcpo's of  $D$  as closed subsets. The  $d$ -topology on  $D$  is finer than the Scott topology. In fact  $D$  is even Hausdorff in the  $d$ -topology: for  $x \not\leq y$  in  $D$ ,  $D \setminus \downarrow y$  and  $\downarrow y$  are disjoint open sets in the  $d$ -topology, containing  $x$  and  $y$  respectively. Functions that are continuous between dcpo's equipped with the  $d$ -topology are called  $d$ -continuous functions. Scott-continuous functions between dcpo's are  $d$ -continuous [24, Lemma 5].

Recall that  $\mathcal{M}D$  is the smallest sub-dcpo of  $\mathcal{V}D$  that contains  $\mathcal{S}D$ , hence  $\mathcal{M}D$  is actually the topological closure of  $\mathcal{S}D$  in  $\mathcal{V}D$  equipped with the  $d$ -topology. Hence we also say that  $\mathcal{M}D$  is the  $d$ -closure of  $\mathcal{S}D$  inside  $\mathcal{V}D$ .

Let  $f: D \rightarrow [0, 1]$  be a Scott-continuous function and  $\nu \in \mathcal{V}D$ . The integral  $\int_{x \in D} f(x) d\nu$ , defined as the Riemann integral  $\int_0^1 \nu(f^{-1}((t, 1])) dt$ , satisfies the following properties, which can be found in [11].

**Proposition 67.** *Let  $D$  be a dcpo,  $f: D \rightarrow [0, 1]$  be a Scott-continuous function. Then we have the following:*

- 1) *The map  $(\nu_i \mapsto \sum_{i=1}^n r_i \nu_i): \mathcal{V}D \rightarrow \mathcal{V}D$  is Scott-continuous hence  $d$ -continuous, for fixed  $\nu_j, j \neq i$  and  $r_i, i = 1, \dots, n$  with  $\sum_{i=1}^n r_i \leq 1$ .*
- 2) *For  $\sum_{i=1}^n r_i \nu_i \in \mathcal{V}D$ , it is true that  $\int f d \sum_{i=1}^n r_i \nu_i = \sum_{i=1}^n r_i \int f d \nu_i$ .*
- 3) *For  $\nu \in \mathcal{V}D$  and  $f, g \in [D \rightarrow [0, 1]]$ ,  $\int r f + s g d \nu = r \int f d \nu + s \int g d \nu$  for  $r + s \leq 1$ .*

**Proof of Lemma 5 .** We prove the case  $n = 2$  and the general case can be proved similarly. We realize that for a fixed simple valuation  $s \in \mathcal{S}D$ , the map  $(\nu \mapsto r_1 \nu + r_2 s): \mathcal{V}D \rightarrow \mathcal{V}D$  maps  $\mathcal{S}D$  into  $\mathcal{S}D$ . From the previous proposition, Item 1, this map is  $d$ -continuous, it then maps the dcpo-closure of  $\mathcal{S}D$ , which is  $\mathcal{M}D$ , into  $\mathcal{M}D$ , the dcpo-closure of  $\mathcal{S}D$ . That is, for each simple valuation  $s$  and each  $\nu \in \mathcal{M}D$ ,  $r_1 \nu + r_2 s \in \mathcal{M}D$ . Now we fix  $\nu \in \mathcal{M}D$ . Then the map  $\xi \mapsto r_1 \nu + r_2 \xi: \mathcal{V}D \rightarrow \mathcal{V}D$  maps  $\mathcal{S}D$  into  $\mathcal{M}D$ , hence it also maps  $\mathcal{M}D$  into  $\mathcal{M}D$  since it is  $d$ -continuous. This means for  $\xi, \nu \in \mathcal{M}D$ ,  $r_1, r_2 \in [0, 1]$  with  $r_1 + r_2 \leq 1$ ,  $r_1 \nu + r_2 \xi \in \mathcal{M}D$ .  $\square$

**Proof of Theorem 19.** To prove this theorem, we first recall two results due to Heckmann [25, Theorem 2.4, Theorem 5.5]. Specifying these results to dcpo  $D$ , it implies that if  $\nu$  is a point-continuous valuation in  $\mathcal{P}D$ , and  $\nu \in \mathcal{O}$  for  $\mathcal{O}$  an open set in  $\mathcal{P}_w D$ , then there exists a simple valuation  $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$  such that  $\sum_{i=1}^n r_i \delta_{x_i} \leq \nu$  and  $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{O}$ .

Now we fix  $\xi \in \mathcal{P}E$  and  $U \in \sigma(D \times E)$ , and consider the functions

$$F: \mathcal{V}_w D \rightarrow [0, \infty]: \nu \mapsto \int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu$$

and

$$G: \mathcal{V}_w D \rightarrow [0, \infty]: \nu \mapsto \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi,$$

where  $[0, \infty]$  is equipped with the Scott topology. We claim that  $F$  and  $G$  are continuous.

The fact that  $F$  is continuous is straightforward from Remark 11. To see that  $G$  is continuous, we assume that  $\int_{y \in E} \int_{x \in D} \chi_U d\nu d\xi > r$  and aim to find an open set  $\mathcal{U}$  of  $\mathcal{V}_w D$  such that  $\nu \in \mathcal{U}$  and for any  $\nu' \in \mathcal{U}$ ,  $\int_{y \in E} \int_{x \in D} \chi_U d\nu' d\xi > r$ . To this end, we note that  $g: E \rightarrow [0, 1]: y \mapsto \int_{x \in D} \chi_U(x, y) d\nu$  is Scott-continuous. Hence  $[g > r] \cap \mathcal{P}E$  is an open subset of  $\mathcal{P}_w E$  that contains  $\xi$ . Applying the aforementioned result we find a simple valuation  $\sum_{i=1}^n r_i \delta_{y_i} \in \mathcal{S}E$  such that  $\sum_{i=1}^n r_i \delta_{y_i} \leq \xi$  and  $\sum_{i=1}^n r_i \delta_{y_i} \in [g > r]$ . This implies that

$$\int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d \sum_{i=1}^n r_i \delta_{y_i} > r.$$

By applying Equation 3, this in turn implies that

$$\sum_{i=1}^n \int_{x \in D} r_i \chi_U(x, y_i) d\nu > r.$$

Obviously, we could find  $s_i \geq 0, i = 1, \dots, n$  such that  $\int_{x \in D} r_i \chi_U(x, y_i) d\nu > s_i$  and  $\sum_{i=1}^n s_i > r$ . Now we let

$$\mathcal{U} = \bigcap_{i=1}^n [r_i \chi_U(x, y_i) > s_i].$$

By Remark 11 the set  $\mathcal{U}$  is open in  $\mathcal{V}_w D$  and obviously  $\nu \in \mathcal{U}$ . Moreover, for any  $\nu' \in \mathcal{U}$ , we have

$$\int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu' d\xi \geq \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu' d \sum_{i=1}^n r_i \delta_{y_i} = \sum_{i=1}^n \int_{x \in D} r_i \chi_U(x, y_i) d\nu' \geq \sum_{i=1}^n s_i > r.$$

Hence  $G$  is continuous indeed.

The functions  $F$  and  $G$  are also linear from Proposition 67, Item 2. Hence both  $F$  and  $G$  are continuous linear map from  $\mathcal{V}_w D$  to  $[0, \infty]$ , we now apply a varied version of the Schröder-Simpson Theorem, which can be found in [26, Corollary 2.5], to see that  $F$  and  $G$  are uniquely determined by their actions on Dirac measures  $\delta_a, a \in D$ . However, we note that  $F(\delta_a) = \int_{y \in E} \chi_U(a, y) d\xi = G(\delta_a)$ , again by Equation 3. Hence  $F = G$ , and we finish the proof by letting  $\xi$  range in  $\mathcal{P}_w E$ .  $\square$

**Proof of Theorem 20.** We only need to prove that the strength of  $\mathcal{V}_{\mathbf{K}, \leq}$  exists, and is of the same form as  $\tau^\mathcal{V}$ , the strength of  $\mathcal{V}$ , and then conclude with Theorem 19.

We know that for each  $\mathbf{K}$ -category  $\mathbf{K} \subseteq \mathbf{D}$ ,  $\mathcal{V}_{\mathbf{K}, \leq}$  is a monad on **DCPO**. Hence, for any dcpo's  $D$  and  $E$ , and any Scott-continuous map  $f: D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$ , the function

$$f^\dagger: \mathcal{V}_{\mathbf{K}, \leq} D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E: \nu \mapsto \lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu$$

is a well-defined Scott-continuous map.

Now we apply this fact to the map  $g: E \rightarrow \mathcal{V}_{\mathbf{K}, \leq}(D \times E): y \mapsto \delta_{(a, y)}$ , where  $a$  is any fixed element in  $D$ . The map  $g$  is obviously Scott-continuous. Hence for any  $\nu \in \mathcal{V}_{\mathbf{K}, \leq} E$ ,

$$g^\dagger(\nu) = \lambda U \in \sigma(D \times E). \int_{y \in E} \delta_{(a, y)}(U) d\nu = \lambda U \in \sigma(D \times E). \int_{y \in E} \chi_U(a, y) d\nu$$

is in  $\mathcal{V}_{\mathbf{K}, \leq}(D \times E)$ . This implies the map

$$\tau_{D, E}: D \times \mathcal{V}_{\mathbf{K}, \leq} E \rightarrow \mathcal{V}_{\mathbf{K}, \leq}(D \times E): (a, \nu) \mapsto \lambda U \in \sigma(D \times E). \int_{y \in E} \chi_U(a, y) d\nu$$

is well-defined, and it is obviously Scott-continuous. Note that apart from the domain and codomain, the map  $\tau_{D, E}$  is same to the strength  $\tau_{D, E}^\mathcal{V}$  of  $\mathcal{V}$  at  $(D, E)$ . Then the same arguments as in Jones' thesis would show that  $\tau_{D, E}$  is the strength of  $\mathcal{V}_{\mathbf{K}, \leq}$  at  $(D, E)$ . Hence  $\mathcal{V}_{\mathbf{K}, \leq}$  is a strong monad.  $\square$

**Proof of Proposition 31.** We first prove that  $K$  is a pointed barycentric algebra. It is easy to see that  $\beta(\mathbf{0}_K)$  is the least element in  $K$ , since for any  $x \in K$ ,  $\beta(\mathbf{0}_K) \leq \beta(\delta_x) = x$ . It is also easy to see that  $a +_1 b = a$ ,  $a +_r b = b +_{1-r} a$  and  $a +_r a = a$ . We now proceed to prove that  $(a +_p b) +_r c = a +_{pr} (b +_{\frac{r-pr}{1-pr}} c)$  for any  $r, p < 1$  and  $a, b, c \in K$ . To this end, we perform the following:

$$\begin{aligned} (a +_p b) +_r c &= \beta(\delta_{a+_p b} +_r \delta_c) && \text{definition of } +_r \\ &= \beta(\delta_{\beta(\delta_{a+_p b})} +_r \delta_{\beta(\delta_c)}) && \text{definition of } +_p \text{ and } \beta(\delta_c) = c \\ &= \beta(\mathcal{M}(\beta)(\delta_{\delta_{a+_p b}} +_r \delta_{\delta_c})) && \mathcal{M}(\beta) \text{ is linear and } \mathcal{M}(\beta)(\delta_\nu) = \delta_{\beta(\nu)} \\ &= \beta(\mu_K^\mathcal{M}(\delta_{\delta_{a+_p b}} +_r \delta_{\delta_c})) && (K, \beta) \text{ is an } \mathcal{M}\text{-algebra} \\ &= \beta((\delta_a +_p \delta_b) +_r \delta_c) && \mu_K^\mathcal{M} \text{ is the multiplication of } \mathcal{M} \text{ at } K \\ &= \beta(\delta_a +_{pr} (\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)) && MK \text{ is a Kegelspitze} \\ &= \beta(\mu_K^\mathcal{M}(\delta_{\delta_a} +_{pr} \delta_{(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)})) && \mu_K^\mathcal{M} \text{ is the multiplication of } \mathcal{M} \text{ at } K \\ &= \beta(\mathcal{M}(\beta)(\delta_{\delta_a} +_{pr} \delta_{(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)})) && (K, \beta) \text{ is an } \mathcal{M}\text{-algebra} \\ &= \beta(\delta_{\beta(\delta_a)} +_{pr} \delta_{\beta(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)}) && \mathcal{M}(\beta) \text{ is linear and } \mathcal{M}(\beta)(\delta_\nu) = \delta_{\beta(\nu)} \\ &= \beta(\delta_a +_{pr} \delta_{(b +_{\frac{r-pr}{1-pr}} c)}) && \text{definition of } b +_{\frac{r-pr}{1-pr}} c \text{ and } \beta(\delta_a) = a \\ &= a +_{pr} (b +_{\frac{r-pr}{1-pr}} c). && \text{definition of } +_{pr} \end{aligned}$$

The map  $(a, b) \mapsto a +_r b = \beta(\delta_a +_r \delta_b): K \times K \rightarrow K$  is Scott-continuous since  $\beta$  and  $\delta$  are Scott-continuous and  $MK$  is a Kegelspitze. The map  $(r, a) \mapsto ra = a +_r \beta(\mathbf{0}_K) = \beta(\delta_a +_r \delta_{\beta(\mathbf{0}_K)}): [0, 1] \times K \rightarrow K$  is Scott-continuous in  $a$  for the exactly same reasons; to see that it also is Scott-continuous in  $r$ , we only need to show that  $r \mapsto \delta_a +_r \delta_{\beta(\mathbf{0}_K)}: [0, 1] \rightarrow MK$  is Scott-continuous for any fixed  $a \in K$ . This is true if  $\beta(\mathbf{0}_K) \leq a$ . However, we already see that  $\beta(\mathbf{0}_K)$  is the least element in  $K$ . Hence we have proved that  $K$  is a Kegelspitze. The map  $\beta$  is clearly linear.  $\square$

**Proof of Proposition 32.**

**The “if” direction:** Assume that  $f: K_1 \rightarrow K_2$  is linear. We need to prove that  $f \circ \beta_1 = \beta_2 \circ \mathcal{M}(f)$ . Since both sides are Scott-continuous hence d-continuous and  $K_2$  is Hausdorff in the d-topology (if  $K_2$  has more than one elements). We only need to prove they are equal on simple valuations on  $K_1$ . To this end, we pick  $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{MK}_1$ , and see

$$\begin{aligned}
 f(\beta_1(\sum_{i=1}^n r_i \delta_{x_i})) &= f(\sum_{i=1}^n r_i x_i) && \beta_1 \text{ is linear and } \beta_1(\delta_{x_i}) = x_i \\
 &= \sum_{i=1}^n r_i f(x_i) && f \text{ is linear} \\
 &= \beta_2(\sum_{i=1}^n r_i \delta_{f(x_i)}) && \beta_2 \text{ is linear and } \beta_2(\delta_{f(x_i)}) = f(x_i) \\
 &= \beta_2(\mathcal{M}(f)(\sum_{i=1}^n r_i \delta_{x_i})). && \mathcal{M}(f) \text{ is linear and } \mathcal{M}(f)(\delta_{x_i}) = \delta_{f(x_i)}
 \end{aligned}$$

**The “only if” direction:** Assume that  $f: K_1 \rightarrow K_2$  is an algebra morphism from  $(K_1, \beta_1)$  to  $(K_2, \beta_2)$ . Then we know that  $f \circ \beta_1 = \beta_2 \circ \mathcal{M}(f)$ . We prove that  $f$  is linear. First, for  $a, b \in K_1$  and  $r \in [0, 1]$ , we have

$$\begin{aligned}
 f(a +_r b) &= f(\beta_1(\delta_a +_r \delta_b)) && \text{definition of } a +_r b \\
 &= \beta_2(\mathcal{M}(f)(\delta_a +_r \delta_b)) && f \text{ is an algebra morphism} \\
 &= \beta_2(\delta_{f(a)} +_r \delta_{f(b)}) && \mathcal{M}(f) \text{ is linear and } \mathcal{M}(f)(\delta_x) = \delta_{f(x)} \\
 &= f(a) +_r f(b). && \text{definition of } f(a) +_r f(b)
 \end{aligned}$$

Second, to prove that  $f$  maps  $\beta_1(\mathbf{0}_{K_1})$  to  $\beta_2(\mathbf{0}_{K_2})$ , we see that  $f(\beta_1(\mathbf{0}_{K_1})) = \beta_2(\mathcal{M}(f)(\mathbf{0}_{K_1})) = \beta_2(\mathbf{0}_{K_2})$  because  $\mathcal{M}(f)$  is linear. □

APPENDIX B  
SOLVING RECURSIVE DOMAIN EQUATIONS IN  $\mathbf{DCPO}_{\mathcal{M}}$

We use  $(\mathcal{M}, \eta, \mu, \tau)$  to indicate our commutative monad and we write  $(\mathcal{L}, \eta^{\mathcal{L}}, \mu^{\mathcal{L}}, \tau^{\mathcal{L}})$  to indicate the lift monad on  $\mathbf{DCPO}$ , which is also commutative.

Recall that the lift monad  $\mathcal{L} : \mathbf{DCPO} \rightarrow \mathbf{DCPO}$  freely adds a new least element, often denoted  $\perp$ , to a dcpo  $X$ . The resulting dcpo is  $\mathcal{L}X \stackrel{\text{def}}{=} X_{\perp}$ . The monad structure of  $\mathcal{L}$  is defined by the following assignments:

$$\begin{array}{ccc} \eta_X^{\mathcal{L}} : X \rightarrow X_{\perp} & \mu_X^{\mathcal{L}} : (X_{\perp_1})_{\perp_2} \rightarrow X_{\perp} & \tau_{XY}^{\mathcal{L}} : X_{\perp} \times Y \rightarrow (X \times Y)_{\perp} \\ x \mapsto x & x \mapsto \begin{cases} \perp & , \text{ if } x = \perp_1 \\ \perp & , \text{ if } x = \perp_2 \\ x & , \text{ if } \perp_1 \neq x \neq \perp_2 \end{cases} & (x, y) \mapsto \begin{cases} \perp & , \text{ if } x = \perp \\ (x, y) & , \text{ if } x \neq \perp \end{cases} \end{array}$$

We write  $\mathbf{DCPO}_{\mathcal{L}}$  for the Kleisli category of  $\mathcal{L}$  and we write its morphisms as  $f : X \rightarrow Y$ , which is by definition a morphism  $f : X \rightarrow Y_{\perp}$  in  $\mathbf{DCPO}$ . We write  $X \otimes Y$  and  $X \oplus Y$  for the symmetric monoidal product and coproduct, respectively, which are (canonically) induced by the commutative monad  $\mathcal{L}$ .

**Proposition 68.** *The assignment  $\phi : \mathcal{L} \Rightarrow \mathcal{M}$  defined by*

$$\begin{array}{l} \phi_X : X_{\perp} \rightarrow \mathcal{M}X \\ x \mapsto \begin{cases} \mathbf{0}_X & , \text{ if } x = \perp \\ \delta_x & , \text{ if } x \neq \perp \end{cases} \end{array}$$

is a strong map of monads (see [29, Definition 5.2.9] for more details).

*Proof.* To see that  $\phi$  is a natural transformation, we need to show, for any Scott-continuous map  $f : X \rightarrow Y$ ,  $\phi_Y \circ \mathcal{L}f = \mathcal{M}f \circ \phi_X : X_{\perp} \rightarrow \mathcal{M}Y$ . However, it is easy to see that both sides send  $\perp$  to  $\mathbf{0}_Y$  and  $x$  that is not  $\perp$  to  $\delta_{f(x)}$ .

Now, we first verify that  $\phi$  is a map of monads. That is, for each dcpo  $X$ , we need to prove that  $\phi_X \circ \eta_X^{\mathcal{L}} = \eta_X$  and  $\phi_X \circ \mu_X^{\mathcal{L}} = \mu_X \circ \mathcal{M}(\phi_X) \circ \phi_{X_{\perp}} : (X_{\perp_1})_{\perp_2} \rightarrow \mathcal{M}(X)$ .

The first equation is trivial, hence we proceed to prove the second. For this, we see

$$\phi_X \circ \mu_X^{\mathcal{L}}(x) = \begin{cases} \phi_X(\perp) = \mathbf{0}_X & , \text{ if } x = \perp_1 \text{ or } x = \perp_2 \\ \phi_X(x) = \delta_x & , \text{ if } \perp_1 \neq x \neq \perp_2 \end{cases}$$

and

$$\mu_X \circ \mathcal{M}(\phi_X) \circ \phi_{X_{\perp}}(x) = \begin{cases} \mu_X \circ \mathcal{M}(\phi_X)(\mathbf{0}_{X_{\perp}}) = \mu_X(\mathbf{0}_{\mathcal{M}X}) = \mathbf{0}_X & , \text{ if } x = \perp_2 \\ \mu_X \circ \mathcal{M}(\phi_X)(\delta_{\perp}) = \mu_X(\delta_{\phi_X(\perp)}) = \mu_X(\delta_{\mathbf{0}_X}) = \mathbf{0}_X & , \text{ if } x = \perp_1 \\ \mu_X \circ \mathcal{M}(\phi_X)(\delta_x) = \mu_X(\delta_{\phi_X(x)}) = \mu_X(\delta_{\delta_x}) = \delta_x & , \text{ if } \perp_1 \neq x \neq \perp_2 . \end{cases}$$

Hence  $\phi : \mathcal{L} \Rightarrow \mathcal{M}$  is a map of monads.

To prove that  $\phi$  is a strong map of monads, we need to show that for any dcpo's  $X$  and  $Y$ ,

$$\tau_{XY} \circ (\phi_X \times \text{id}_Y) = \phi_{XY} \circ \tau_{XY}^{\mathcal{L}} : X_{\perp} \times Y \rightarrow \mathcal{M}(X \times Y).$$

The strength  $\tau$  of  $\mathcal{M}$  at  $(X, Y)$  is defined as follows:

$$\tau_{XY} : \mathcal{M}X \times Y \rightarrow \mathcal{M}(X \times Y) : (\nu, y) \mapsto \lambda U. \int_{x \in X} \chi_U(x, y) d\nu,$$

where  $\chi_U$  is the characteristic function of  $U \in \sigma(X \times Y)$ , i.e.,  $\chi_U(x, y) = 1$  if  $(x, y) \in U$  and  $\chi_U(x, y) = 0$ , otherwise. Now we perform the following computation

$$\tau_{XY} \circ (\phi_X \times \text{id}_Y)(x, y) = \begin{cases} \tau_{XY}(\mathbf{0}_X, y) = \lambda U. \int_{x \in X} \chi_U(x, y) d\mathbf{0}_X = \lambda U. 0 = \mathbf{0}_{X \times Y} & , \text{ if } x = \perp \\ \tau_{XY}(\delta_x, y) = \lambda U. \int_{x \in X} \chi_U(x, y) d\delta_x = \lambda U. \chi_U(x, y) = \delta_{(x, y)} & , \text{ if } x \neq \perp \end{cases}$$

and

$$\phi_{XY} \circ \tau_{XY}^{\mathcal{L}}(x, y) = \begin{cases} \phi_{XY}(\perp) = \mathbf{0}_{X \times Y} & , \text{ if } x = \perp \\ \phi_{XY}((x, y)) = \delta_{(x, y)} & , \text{ if } x \neq \perp \end{cases}$$

which concludes the proof. □

Recall that any map of monads induces a functor between the corresponding Kleisli categories of the two monads (see [29, Exercise 5.2.1]). This allows us to show the next corollary.



**Corollary 69.** *The functor  $\mathcal{F} : \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ , induced by  $\phi : \mathcal{L} \Rightarrow \mathcal{M}$ , and defined by:*

$$\begin{aligned}\mathcal{F}X &\stackrel{\text{def}}{=} X \\ \mathcal{F}(f : X \rightarrow Y) &\stackrel{\text{def}}{=} \phi_Y \circ f\end{aligned}$$

*strictly preserves the monoidal and coproduct structures in the sense that the following equalities:*

$$\begin{aligned}\mathcal{F}(X \otimes Y) &= \mathcal{F}X \dot{\times} \mathcal{F}Y & \mathcal{F}(X \oplus Y) &= \mathcal{F}X \dot{+} \mathcal{F}Y \\ \mathcal{F}(f \otimes g) &= \mathcal{F}f \dot{\times} \mathcal{F}g & \mathcal{F}(f \oplus g) &= \mathcal{F}f \dot{+} \mathcal{F}g\end{aligned}$$

*hold.*

*Proof.* This follows by canonical categorical arguments and is just a straightforward verification.  $\square$

Before we may prove our next proposition, let us recall an important result from [33].

**Proposition 70.** *Let  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  be  $\mathbf{DCPO}$ -enriched categories. Assume further that  $\mathbf{A}$  and  $\mathbf{B}$  have all  $\omega$ -colimits (or all  $\omega^{\text{op}}$ -limits). If  $\mathcal{T} : \mathbf{A}^{\text{op}} \times \mathbf{B} \rightarrow \mathbf{C}$  is a  $\mathbf{DCPO}$ -enriched functor, then the assignment*

$$\begin{aligned}\mathcal{T}^E : \mathbf{A}_e \times \mathbf{B}_e &\rightarrow \mathbf{C}_e \\ \mathcal{T}^E(A, B) &\stackrel{\text{def}}{=} \mathcal{T}(A, B) \\ \mathcal{T}^E(e_1, e_2) &\stackrel{\text{def}}{=} \mathcal{T}(e_1^p, e_2)\end{aligned}$$

*defines a covariant  $\omega$ -cocontinuous functor.*

*Proof.* This follows by combining several results from [33], namely Theorem 2, the corollary after it and Theorem 3.  $\square$

Therefore, by trivialising the category  $\mathbf{A}$ , we may obtain results for purely covariant functors. When neither category is trivialised, this allows us to interpret mixed-variance functors (such as function space) as covariant functors on subcategories of embeddings.

**Proposition 71.** *The category  $\mathbf{PD}_e$  has an initial object and all  $\omega$ -colimits and the following assignments:*

$$\begin{aligned}\dot{\times}_e : \mathbf{PD}_e \times \mathbf{PD}_e &\rightarrow \mathbf{PD}_e & \dot{+}_e : \mathbf{PD}_e \times \mathbf{PD}_e &\rightarrow \mathbf{PD}_e \\ X \dot{\times}_e Y &\stackrel{\text{def}}{=} X \dot{\times} Y & X \dot{+}_e Y &\stackrel{\text{def}}{=} X \dot{+} Y \\ e_1 \dot{\times}_e e_2 &\stackrel{\text{def}}{=} e_1 \dot{\times} e_2 & e_1 \dot{+}_e e_2 &\stackrel{\text{def}}{=} e_1 \dot{+} e_2\end{aligned}$$

$$\begin{aligned}[\rightarrow]_e^{\mathcal{J}} : \mathbf{PD}_e \times \mathbf{PD}_e &\rightarrow \mathbf{PD}_e \\ [X \rightarrow Y]_e^{\mathcal{J}} &\stackrel{\text{def}}{=} \mathcal{J}[X \rightarrow Y] \\ [e_1 \rightarrow e_2]_e^{\mathcal{J}} &\stackrel{\text{def}}{=} \mathcal{J}[e_1^p \rightarrow e_2]\end{aligned}$$

*define covariant  $\omega$ -cocontinuous functors on  $\mathbf{PD}_e$ .*

*Proof.* The empty dcpo  $\emptyset$  is a zero object in  $\mathbf{PD}$  such that each map  $e : \emptyset \rightarrow X$  is an embedding and each map  $p : X \rightarrow \emptyset$  is a projection. Therefore,  $\emptyset$  is initial in  $\mathbf{PD}_e$ . The existence of all  $\omega$ -colimits in  $\mathbf{PD}_e$  follows from the existence of all  $\omega$ -colimits of  $\mathbf{PD}$  together with results from [33].

Next, we show that  $\dot{\times} : \mathbf{DCPO}_{\mathcal{M}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  restricts to a functor  $\dot{\times}^{\mathbf{PD}} : \mathbf{PD} \times \mathbf{PD} \rightarrow \mathbf{PD}$ . On objects, this is obvious. For morphisms, observe that the morphisms of  $\mathbf{PD}$  are exactly those which are in the image of  $\mathcal{F}$ . Therefore  $\dot{\times}^{\mathbf{PD}}$  restricts as indicated because  $\mathcal{F}f \dot{\times} \mathcal{F}g = \mathcal{F}(f \otimes g)$  by Corollary 69. Then, by Proposition 70, it follows that  $(\dot{\times}^{\mathbf{PD}})^E : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$  is a covariant  $\omega$ -cocontinuous functor. However, by definition,  $\dot{\times}_e = (\dot{\times}^{\mathbf{PD}})^E$  which shows the result for  $\dot{\times}_e$ .

Exactly the same argument (swapping  $\dot{\times}$  for  $\dot{+}$  and  $\otimes$  for  $\oplus$ ) shows the result for  $\dot{+}_e$ .

For function spaces, consider the functor  $\mathcal{J} \circ [\rightarrow] : \mathbf{DCPO}_{\mathcal{M}}^{\text{op}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ . This composition (co)restricts to a functor  $(\mathcal{J} \circ [\rightarrow])^{\mathbf{PD}} : \mathbf{PD}^{\text{op}} \times \mathbf{PD} \rightarrow \mathbf{PD}$ , because  $\mathcal{J}(f \rightarrow g) = \eta \circ (f \rightarrow g) = \phi \circ \eta^{\mathcal{L}} \circ (f \rightarrow g) = \mathcal{F}(\eta^{\mathcal{L}} \circ (f \rightarrow g))$ . By Proposition 70, it follows  $((\mathcal{J} \circ [\rightarrow])^{\mathbf{PD}})^E : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$  is a covariant  $\omega$ -cocontinuous functor. Finally, by definition,  $[\rightarrow]_e^{\mathcal{J}} = ((\mathcal{J} \circ [\rightarrow])^{\mathbf{PD}})^E$  which concludes the proof.  $\square$

We conclude the appendix with a proof that the subcategories  $\mathbf{TD}$  and  $\mathbf{PD}$  contain the same isomorphisms.

**Proposition 72.** *Every isomorphism of  $\mathbf{PD}$  is also an isomorphism of  $\mathbf{TD}$ .*

*Proof.* Observe that, by definition, the morphisms of **TD** are those in the image of  $\mathcal{J} : \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  and the morphisms of **PD** are those in the image of  $\mathcal{F} : \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ . Then, it is easy to see that the following diagram:

$$\begin{array}{ccc}
 \mathbf{TD} & \hookrightarrow & \mathbf{PD} \\
 \cong & & \cong \\
 \mathbf{DCPO} & \xrightarrow{\mathcal{J}^{\mathcal{L}}} & \mathbf{DCPO}_{\mathcal{L}}
 \end{array}$$

commutes, where:

- the top arrow is the subcategory inclusion  $\mathbf{TD} \hookrightarrow \mathbf{PD}$ ;
- the left vertical isomorphism is the corestriction of  $\mathcal{J}$  to **TD**;
- the right vertical isomorphism is the corestriction of  $\mathcal{F}$  to **PD**;
- the functor  $\mathcal{J}^{\mathcal{L}}$  is the Kleisli inclusion of **DCPO** into **DCPO** <sub>$\mathcal{L}$</sub> , defined by  $\mathcal{J}^{\mathcal{L}}(X) \stackrel{\text{def}}{=} X$  and  $\mathcal{J}^{\mathcal{L}}(f) \stackrel{\text{def}}{=} \eta^{\mathcal{L}} \circ f$ .

It is well-known (and easy to prove) that if  $f : X \rightarrow Y$  in **DCPO** <sub>$\mathcal{L}$</sub>  is an isomorphism, then there exists  $f' : X \rightarrow Y$  in **DCPO** which is also an isomorphism and  $f = \mathcal{J}^{\mathcal{L}}(f')$ . The proof is finished by a simple diagram chase using this fact.  $\square$

APPENDIX C

PRODUCTS, COPRODUCTS AND KLEISLI COMPOSITION PRESERVE BARYCENTRIC SUMS OF FUNCTIONS

The monoidal product  $_ \dot{\times} _ : \mathbf{DCPO}_{\mathcal{M}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  is defined as: for dcpo's  $A$  and  $B$ ,  $A \dot{\times} B \stackrel{\text{def}}{=} A \times B$ , and for Scott-continuous maps  $f : A \rightarrow \mathcal{M}C$  and  $g : B \rightarrow \mathcal{M}D$ ,  $f \dot{\times} g \stackrel{\text{def}}{=} \lambda(a, b). f(a) \otimes g(b)$ , where  $f(a) \otimes g(b)$  is defined in Remark 8. For  $f, h : A \rightarrow \mathcal{M}C$  and  $r \in [0, 1]$ ,  $f +_r h$  is defined pointwise, that is,  $(f +_r h)(a) = f(a) +_r h(a) = rf(a) + (1-r)h(a)$ . It follows from Lemma 5 that  $f +_r h$  is well-defined and obviously  $f +_r h$  is Scott-continuous, hence  $f +_r h \in [A \rightarrow \mathcal{M}C]$ .

**Proposition 73.** For  $f, h : A \rightarrow \mathcal{M}C$ ,  $g : B \rightarrow \mathcal{M}D$  and  $r \in [0, 1]$ , we have

- 1)  $(f +_r h) \dot{\times} g = f \dot{\times} g +_r h \dot{\times} g : A \dot{\times} B \rightarrow \mathcal{M}(C \dot{\times} D)$ ;
- 2)  $g \dot{\times} (f +_r h) = g \dot{\times} f +_r g \dot{\times} h : B \dot{\times} A \rightarrow \mathcal{M}(D \dot{\times} C)$ .

*Proof.* We only prove Item 1, the second item can be proved similarly.

For each  $(a, b) \in A \dot{\times} B$ , we have the following:

$$\begin{aligned}
 & ((f +_r h) \dot{\times} g)(a, b) \\
 &= (f +_r h)(a) \otimes g(b) && \text{definition of } _ \dot{\times} _ \\
 &= (f(a) +_r h(a)) \otimes g(b) && \text{definition of } f +_r h \\
 &= \lambda U \in \sigma(C \dot{\times} D). \int_{y \in D} \int_{x \in C} \chi_U(x, y) d(f(a) +_r h(a)) dg(b) && \text{definition of the operation } \otimes \\
 &= \lambda U. \int_{y \in D} \left( \int_{x \in C} \chi_U(x, y) df(a) +_r \int_{x \in C} \chi_U(x, y) dh(a) \right) dg(b) && \text{by Proposition 67, Item 2} \\
 &= \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) df(a) dg(b) +_r \int_{y \in D} \int_{x \in C} \chi_U(x, y) dh(a) dg(b) && \text{by Proposition 67, Item 3} \\
 &= \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) df(a) dg(b) +_r \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) dh(a) dg(b) && \text{by definition of } +_r \text{ of valuations} \\
 &= (f \dot{\times} g)(a, b) +_r (h \dot{\times} g)(a, b) && \text{definition of } _ \dot{\times} _ \\
 &= (f \dot{\times} g +_r h \dot{\times} g)(a, b) && \text{definition of } +_r \text{ of functions.}
 \end{aligned}$$

Hence the proof is completed.  $\square$

The functor  $_ \dot{+} _ : \mathbf{DCPO}_{\mathcal{M}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$  is defined as: for dcpo's  $A$  and  $B$ ,  $A \dot{+} B \stackrel{\text{def}}{=} A + B$ , and for Scott-continuous maps  $f : A \rightarrow \mathcal{M}C$  and  $g : B \rightarrow \mathcal{M}D$ ,  $f \dot{+} g = [\mathcal{M}(i_C) \circ f, \mathcal{M}(i_D) \circ g]$ , where  $i_C : C \rightarrow C + D$  and  $i_D : D \rightarrow C + D$  are the obvious injections.

**Proposition 74.** For  $f, h : A \rightarrow \mathcal{M}C$ ,  $g : B \rightarrow \mathcal{M}D$  and  $r \in [0, 1]$ , we have

- 1)  $(f +_r h) \dot{+} g = (f \dot{+} g) +_r (h \dot{+} g)$ ;
- 2)  $g \dot{+} (f +_r h) = (g \dot{+} f) +_r (g \dot{+} h)$ .

*Proof.* Again, we only prove the first claim as the second can be proved similarly. Let  $a \in A$ , we perform the following computation:

$$\begin{aligned}
 ((f +_r h) \dot{+} g)(i_A(a)) &= [\mathcal{M}(i_C) \circ (f +_r h), \mathcal{M}(i_D) \circ g](i_A(a)) && \text{definition of } _ \dot{+} _ \\
 &= \mathcal{M}(i_C)((f +_r h)(a)) && \text{obvious} \\
 &= \mathcal{M}(i_C)(f(a) +_r h(a)) && \text{definition of } f +_r h \\
 &= \lambda U. (f(a) +_r h(a))(i_C^{-1}(U)) && \text{definition of } \mathcal{M}(i_C) \\
 &= \lambda U. f(a)(i_C^{-1}(U)) +_r h(a)(i_C^{-1}(U)) && \text{definition of } f(a) +_r h(a) \\
 &= \lambda U. f(a)(i_C^{-1}(U)) +_r \lambda U. h(a)(i_C^{-1}(U)) && \text{definition of } +_r \text{ of valuations} \\
 &= \mathcal{M}(i_C)(f(a)) +_r \mathcal{M}(i_C)(h(a)), && \text{definition of } \mathcal{M}(i_C) \\
 &= (f \dot{+} g)(i_A(a)) +_r (h \dot{+} g)(i_A(a)) && \text{definition of } _ \dot{+} _ \\
 &= ((f \dot{+} g) +_r (h \dot{+} g))(i_A(a)). && \text{definition of } (f \dot{+} g) +_r (h \dot{+} g)
 \end{aligned}$$

Moreover, it is easy to see that for  $b \in B$ ,  $((f +_r h) \dot{+} g)(i_B(b)) = \mathcal{M}(i_D)(g(b)) = \mathcal{M}(i_D)(g(b)) +_r \mathcal{M}(i_D)(g(b)) = ((f \dot{+} g) +_r (h \dot{+} g))(i_B(b))$ . Hence we finish the proof.  $\square$

Recall that in  $\mathbf{DCPO}_{\mathcal{M}}$  the Kleisli composition  $\circ : [A \rightarrow B] \times [B \rightarrow C] \rightarrow [A \rightarrow C]$  is given by

$$(f, g) \mapsto g \circ f = g^{\ddagger} \circ f.$$

**Proposition 75.** For  $f, h: A \rightarrow \mathcal{M}B, g, k: B \rightarrow \mathcal{M}C$  and  $r \in [0, 1]$ , we have

- 1)  $g \circ (f +_r h) = g \circ f +_r g \circ h;$
- 2)  $(g +_r k) \circ f = g \circ f +_r k \circ f.$

*Proof.* 1) Let  $a \in A$ . We have

$$\begin{aligned}
 g \circ (f +_r h)(a) &= (g^\ddagger \circ (f +_r h))(a) && \text{definition of } \circ \\
 &= g^\ddagger(f(a) +_r h(a)) && \text{definition of } f +_r h \\
 &= \lambda U. \int_{x \in B} g(x)(U) d(f(a) +_r h(a)) && \text{definition of } g^\ddagger \\
 &= \lambda U. \int_{x \in B} g(x)(U) df(a) +_r \lambda U. \int_{x \in B} g(x)(U) dh(a) && \text{by Proposition 67, Item 2} \\
 &= g^\ddagger(f(a)) +_r g^\ddagger(h(a)) && \text{definition of } g^\ddagger \\
 &= (g \circ f +_r g \circ h)(a).
 \end{aligned}$$

2) Let  $a \in A$ . We have

$$\begin{aligned}
 ((g +_r k) \circ f)(a) &= (g +_r k)^\ddagger(f(a)) && \text{definition of } \circ \\
 &= \lambda U. \int_{x \in B} (g +_r k)(x)(U) df(a) && \text{definition of } _\ddagger \\
 &= \lambda U. \int_{x \in B} g(x)(U) df(a) +_r \lambda U. \int_{x \in B} k(x)(U) df(a) && \text{by Proposition 67, Item 3} \\
 &= g^\ddagger(f(a)) +_r k^\ddagger(f(a)) && \text{definition of } _\ddagger \\
 &= (g \circ f +_r k \circ f)(a).
 \end{aligned}$$

□

APPENDIX D  
ENRICHING HEREDITARILY ATOMIC VON NEUMANN ALGEBRAS

Thus we have to show that  $\mathbf{HA}(M, N)$  is a continuous dcpo for each  $M, N \in \mathbf{HA}$ . We will rely heavily the following lemma, which follows from [44, Example 2.7]:

**Lemma 76.** *Let  $M$  be a von Neumann algebra. If  $M$  is hereditarily atomic, then  $[0, 1]_M$  is a continuous dcpo.*

We note that the converse of this lemma is shown in [45].

Given a von Neumann algebra  $M$  and a Hilbert space  $H$ , any completely positive map  $\varphi : M \rightarrow B(H)$  can be decomposed as  $\varphi(x) = v^* \pi(x) v$ , where  $\pi : M \rightarrow B(K)$  is a *representation* of  $M$  on some Hilbert space  $K$ , i.e., a unital  $*$ -homomorphism, and  $v : H \rightarrow K$  is a linear map such that  $\|v\|^2 \leq \|\varphi(1)\|$ . We say that  $(\pi, v, K)$  is a *Stinespring representation* of  $\varphi$ . Moreover,  $K$  can be chosen to be *minimal*, i.e.,  $\pi[M]vH$  is norm dense in  $K$ ; up to unitary equivalence, the minimal Stinespring decomposition of  $\varphi$  is unique. A proof of the existence of the minimal Stinespring representation is given in [35, Theorem II.6.9.7] and in [46, Theorem 1.2.7]. In the proof of [35, Theorem III.2.2.4] it is asserted that  $\pi$  is normal when  $\varphi$  is normal.

**Proposition 77.** *Let  $M$  be a von Neumann algebra, let  $H$  be a Hilbert space, and let  $\varphi : M \rightarrow B(H)$  be a normal completely positive map with minimal Stinespring representation  $(\pi, v, K)$ . Then we have an order isomorphism  $[0, 1]_{\pi[M]'} \rightarrow \downarrow \varphi$ ,  $t \mapsto \varphi_t$ , where  $\varphi_t(x) = v^* t \pi(x) v$  for each  $x \in M$  and where the downset is taken in  $\mathbf{vN}(M, B(H))$ .*

*Proof.* It follows from [46, Theorem 3.5.3] and the paragraph preceding it that the assignment  $t \mapsto \varphi_t$  is a bijection from  $[0, 1]_{\pi[M]'}$  to the set of all completely positive maps  $\psi : M \rightarrow B(H)$  such that  $\varphi - \psi$  is completely positive. Hence any such  $\psi$  is of the form  $v^* t \pi v$  for some  $t \in [0, 1]_{\pi[M]'}$ . Since  $\varphi$  is normal, so is  $\pi$ , whence also  $\psi = v^* t \pi v$  is normal. Moreover,  $\varphi$  is subunital, and  $\varphi - \psi$  is completely positive, so positive, hence  $\psi(1_M) \leq \varphi(1_M) \leq 1_H$ , so also  $\psi$  is subunital. Thus the assignment  $t \mapsto \varphi_t$  is indeed a bijection between  $[0, 1]_{\pi[M]'}$  and  $\downarrow \varphi$ . We have to show that it is an order isomorphism. Let  $t_1 \leq t_2$  in  $[0, 1]_{\pi[M]'}$ . Let  $x \in M$  be positive, so  $x = y^* y$  for some  $y \in M$ . Then for each  $i = 1, 2$ , we have that  $\pi(y^*)$  commutes with  $t_i$  (since the latter is an element of  $\pi[M]'$ ), whence  $\varphi_{t_i}(x) = v^* t_i \pi(y^*) \pi(y) v = v^* \pi(y)^* t_i \pi(y) v$ . Since for any operator  $b$  the assignment  $t \mapsto b^* t b$  preserves the order (cf. [37, Corollary 4.2.7]), it now follows that  $\varphi_{t_1} \leq \varphi_{t_2}$ .

Next assume that  $\varphi_{t_1} \leq \varphi_{t_2}$ . We have to show that  $t_1 \leq t_2$ . Write  $\varphi_i := \varphi_{t_i}$  for  $i = 1, 2$ , and  $\varphi_3 := \varphi$ . For each  $i = 1, 2, 3$ , let  $(\pi_i, v_i, K_i)$  be the minimal Stinespring representation of  $\varphi_i$ . For  $i \leq j$  in  $\{1, 2, 3\}$ , [46, Lemma 3.5.2] assures the existence of operators  $s_{ij} : K_j \rightarrow K_i$  with  $\|s_{ij}\| \leq 1$  such that

- (i)  $s_{ij} v_j = v_i$ ;
- (ii)  $s_{ij} \pi_j(x) = \pi_i(x) s_{ij}$  for each  $x \in M$ .

As a consequence, for each  $x \in M$  we have

$$v^* s_{i3}^* s_{i3} \pi(x) v = v_3^* s_{i3}^* s_{i3} \pi_3(x) v_3 = v_i^* s_{i3} \pi_3(x) v_3 = v_i^* \pi_i(x) s_{i3} v_3 = v_i^* \pi_i(x) v_i = \varphi_i(x)$$

and for each  $x \in M$  we have

$$s_{i3}^* s_{i3} \pi(x) = s_{i3}^* s_{i3} \pi_3(x) = s_{i3}^* \pi_i(x) s_{i3} = s_{i3}^* \pi_i(x^*)^* s_{i3} = (\pi_2(x^*) s_{i3})^* s_{i3} = (s_{i3} \pi_3(x^*))^* s_{i3} = \pi_3(x) s_{i3}^* s_{i3},$$

so  $s_{i3}^* s_{i3} \in \pi[M]'$ . Since  $t_i$  is the unique element in  $\pi[M]'$  such that  $\varphi_i(x) = v^* t_i \pi(x) v$ , it follows that  $t_i = s_{i3}^* s_{i3}$ , and in a similar way we find that  $t_1 = s_{23}^* s_{12}^* s_{12} s_{23}$ . Since  $\|s_{ij}\| \leq 1$ , we have  $\|s_{ij}^* s_{ij}\| = \|s_{ij}\|^2 \leq 1$  using the  $C^*$ -identity. It now follows from [37, Proposition 4.2.3(ii)] that  $s_{ij}^* s_{ij} \leq 1$ , so  $s_{23}^* (1 - s_{12}^* s_{12}) s_{23}$  is positive. But

$$s_{23}^* (1 - s_{12}^* s_{12}) s_{23} = s_{23}^* s_{23} - s_{23}^* s_{12}^* s_{12} s_{23} = t_2 - t_1,$$

so also  $t_2 - t_1$  is positive, i.e.,  $t_1 \leq t_2$ . □

**Lemma 78.** *Let  $M = \prod_{\alpha \in A} M_\alpha$  be the product of a collection  $(M_\alpha : \alpha \in A)$  of von Neumann algebras, let  $\pi_\alpha : M_\alpha \rightarrow B(K_\alpha)$  be a representation on some Hilbert space  $K_\alpha$ , and let  $K \stackrel{\text{def}}{=} \bigoplus_{\alpha \in A} K_\alpha$  be the sum of the Hilbert spaces  $K_\alpha$ . Then  $\pi : M \rightarrow B(K)$  defined by  $\pi(x)k = (\pi_\alpha(x_\alpha)k_\alpha)_{\alpha \in A}$  for each  $x = (x_\alpha)_{\alpha \in A}$  in  $M$  and each  $k = (k_\alpha)_{\alpha \in A}$  in  $K$  is a representation of  $M$  on  $K$  such there is an injective  $*$ -homomorphism  $\rho : \pi[M] \rightarrow \prod_{\alpha \in A} B(K_\alpha)$ .*

*Proof.* Clearly  $\pi$  is a  $*$ -homomorphism, hence a representation of  $M$  onto  $K$ . For  $\alpha \in A$ , let  $e_\alpha : K_\alpha \rightarrow K$  be the embedding and  $p_\alpha : K \rightarrow K_\alpha$  be the projection, which are easily seen to each other's adjoints, and to be bounded with norm at most 1. For each  $y \in B(K)$ , and each  $\alpha, \beta \in A$  let  $y_{\alpha\beta} = p_\beta y e_\alpha : K_\alpha \rightarrow K_\beta$ , so  $y_{\alpha\beta} \in B(K_\alpha, K_\beta)$ . Then  $y$  is completely determined by the  $y_{\alpha\beta}$ . Moreover, for  $y, z \in B(K)$  and  $\beta, \gamma \in A$ , multiplication can be described in a similar way to matrix multiplication:  $(yz)_{\beta\gamma} = \sum_{\alpha \in A} y_{\beta\alpha} z_{\alpha\gamma}$ . Moreover, for  $x = (x_\alpha)_{\alpha \in A}$  in  $M$ ,  $\beta, \gamma \in A$  and each  $a \in K_\gamma$  we have

$$(\pi(x))_{\beta\gamma} a = p_\gamma \pi(x) e_\beta(a) = p_\gamma \pi(x) (\delta_{\alpha\beta} a)_{\alpha \in A} = p_\gamma (\pi_\alpha(x_\alpha) \delta_{\alpha\beta} a)_{\alpha \in A} = \pi_\gamma(x_\gamma) (\delta_{\gamma\beta} a) = \delta_{\beta\gamma} \pi_\gamma(x_\gamma) a,$$

hence  $\pi(x)_{\beta\gamma} = \delta_{\beta\gamma}\pi_\gamma(x_\gamma)$ . Let  $D = \{y \in B(K) : y_{\alpha\beta} = 0 \text{ for each } \alpha \neq \beta \text{ in } A\}$ . We define a map  $\rho : D \rightarrow \prod_{\alpha \in A} B(K_\alpha)$  by  $y \mapsto (y_{\alpha\alpha})_{\alpha \in A}$ . In order to show that  $\rho$  is well defined, let  $y \in D$ . Since for each  $\alpha \in A$ , we have  $\|y_{\alpha\alpha}\| = \|p_\alpha y e_\alpha\| \leq \|p_\alpha\| \|y\| \|e_\alpha\| \leq \|y\|$ , since  $\|p_\alpha\|, \|e_\alpha\| \leq 1$ , it follows that  $\sup_{\alpha \in A} \|y_{\alpha\alpha}\| \leq \|y\| < \infty$ , which shows that  $\rho(y)$  is a well-defined element of  $\prod_{\alpha \in A} B(K_\alpha)$ . Clearly  $\rho$  is a  $*$ -homomorphism. It is injective, because each  $y \in B(K)$  is determined by  $y_{\beta\gamma}$  for  $\beta, \gamma \in A$ , and by definition of  $D$ , we have  $y_{\beta\gamma} = 0$  for each  $\beta \neq \gamma$ .

Let  $y \in \pi[M]'$ , i.e.,  $y \in B(K)$  such that  $\pi(x)y = y\pi(x)$  for each  $x = (x_\alpha)_{\alpha \in A}$  in  $M$ . Let  $\beta \neq \gamma$  in  $A$ . Let  $x_\beta = 1_{M_\beta}$  and let  $x_\alpha = 0$  for each  $\alpha \neq \beta$ . Then  $x = (x_\alpha)_{\alpha \in A}$  is an element of  $M$  and since  $y \in \pi[M]'$ , we find

$$\begin{aligned} y_{\beta\gamma} &= \pi_\beta(x_\beta)y_{\beta\gamma} = \pi_\beta(x_\beta)y_{\beta\gamma} = \sum_{\alpha \in A} \delta_{\beta\alpha}\pi_\alpha(x_\alpha)y_{\alpha\gamma} = \sum_{\alpha \in A} \pi(x)_{\beta\alpha}y_{\alpha\gamma} = (\pi(x)y)_{\beta\gamma} \\ &= (y\pi(x))_{\beta\gamma} = \sum_{\alpha \in A} y_{\beta\alpha}\pi(x)_{\alpha\gamma} = \sum_{\alpha \in A} y_{\beta\alpha}\delta_{\alpha\gamma}\pi_\gamma(x_\gamma) = y_{\beta\gamma}\pi_\gamma(x_\gamma) = 0. \end{aligned}$$

Hence  $y \in D$  from which follows that  $\rho$  restricts to an injective  $*$ -homomorphism  $\pi[M]' \rightarrow \prod_{\alpha \in A} B(K_\alpha)$ .  $\square$

**Lemma 79.** *Let  $(M_\alpha)_{\alpha \in A}$  be a collection of von Neumann algebras, and let  $M = \prod_{\alpha \in A} M_\alpha$ , and for  $\beta \in A$ , denote the embedding  $M_\beta \rightarrow M$  by  $\iota_\beta$ . Let  $H$  be a Hilbert space, and let  $\varphi : M \rightarrow B(H)$  be a normal completely positive subunital map. Then for each  $\alpha \in A$ , the map  $\varphi_\alpha := \varphi \circ \iota_\alpha$  is a completely positive map  $M_\alpha \rightarrow B(H)$ , and if  $(\pi_\alpha, v_\alpha, K_\alpha)$  denotes the minimal Stinespring representation corresponding to  $\varphi_\alpha$ , then the representation  $(\pi, v, K)$  constructed in Lemma 78 from the representations  $(\pi_\alpha, v_\alpha, K_\alpha)$  is the minimal Stinespring representation corresponding to  $\varphi$ .*

*Proof.* Let  $\alpha \in A$ . Then  $\iota_\alpha$  is a (non-unital)  $*$ -homomorphism, hence completely positive, whence  $\varphi_\alpha$  is completely positive. The identity element in  $M_\alpha$  corresponds to a projection  $r_\alpha$  in  $M$ , and  $\sum_{\alpha \in A} r_\alpha = 1_M$ , where the convergence is with respect to the strong operator topology. Since multiplication with a fixed element is continuous with respect to the strong operator topology, it follows that  $x = \sum_{\alpha \in A} x p_\alpha$  for each  $x = (x_\alpha)_{\alpha \in A}$  in  $M$ . It is easy to see that  $x p_\alpha = \iota_{\alpha \in A}(x_\alpha)$ . Since convergence with respect to the strong operator topology implies convergence with respect to the weak operator topology, and the latter topology coincides with the  $\sigma$ -weak operator topology on the unit ball of an operator algebra [36, Lemma II.2.5], it follows that  $x = \sum_{\alpha \in A} \iota_\alpha(x_\alpha)$ , where the sum converges with respect to the  $\sigma$ -weak operator topology.

Since  $\varphi$  is normal, so continuous with respect to the  $\sigma$ -weak operator topology, and since we previously found that  $x = \sum_{\alpha \in A} \iota_\alpha(x_\alpha)$  for each  $x = (x_\alpha)_{\alpha \in A}$  in  $M$ , where the sum converges with respect to the  $\sigma$ -weak operator topology, we obtain

$$\varphi(x) = \varphi\left(\sum_{\alpha \in A} \iota_\alpha(x_\alpha)\right) = \sum_{\alpha \in A} \varphi \circ \iota_\alpha(x_\alpha) = \sum_{\alpha \in A} \varphi_\alpha(x_\alpha).$$

For each  $\alpha \in A$ , let  $(\pi_\alpha, v_\alpha, K_\alpha)$  be the minimal Stinespring representation of  $\varphi_\alpha$ , so  $v_\alpha : H \rightarrow K_\alpha$  satisfies  $\|v_\alpha\|^2 \leq \|\varphi_\alpha(1_{M_\alpha})\|$  and  $\varphi_\alpha(x) = v_\alpha^* \pi_\alpha(x) v_\alpha$  for each  $x \in M_\alpha$ , and  $\overline{\pi_\alpha[M_\alpha]v_\alpha H} = K_\alpha$ . Let  $K = \bigoplus_{\alpha \in A} K_\alpha$ . We want to define  $v : H \rightarrow K$  by  $vh = (v_\alpha h)_{\alpha \in A}$ , but we have to show that each  $vh$  is indeed an element of  $K$ , i.e., we have to show that  $\sum_{\alpha \in A} \|v_\alpha h\|^2 < \infty$ . Since each Stinespring representation  $\pi_\alpha$  is unital, and since  $\varphi$  is subunital, we obtain

$$1_H \geq \varphi(1_M) = \sum_{\alpha \in A} \varphi_\alpha(1_{M_\alpha}) = \sum_{\alpha \in A} v_\alpha^* \pi_\alpha(1_{M_\alpha}) v_\alpha = \sum_{\alpha \in A} v_\alpha^* v_\alpha.$$

Hence for each  $h \in H$ , we obtain

$$\sum_{\alpha \in A} \|v_\alpha h\|^2 = \sum_{\alpha \in A} \langle v_\alpha h, v_\alpha h \rangle = \sum_{\alpha \in A} \langle h, v_\alpha^* v_\alpha h \rangle = \left\langle h, \left( \sum_{\alpha \in A} v_\alpha^* v_\alpha \right) h \right\rangle \leq \langle h, h \rangle = \|h\|^2,$$

where the third equality is due to the fact convergence with respect to the  $\sigma$ -weak operator topology implies convergence with respect to the weak operator topology. Thus  $v : H \rightarrow K$  given by  $vh = (v_\alpha h)_{\alpha \in A}$  is indeed a well defined operator.

Then for  $h, k \in H$ , and  $x = (x_\alpha)_{\alpha \in A}$  in  $M$ , we have

$$\begin{aligned} \langle k, v^* \pi(x) v h \rangle_H &= \langle vk, \pi(x) v h \rangle_K = \langle (v_\alpha k)_{\alpha \in A}, (\pi_\alpha(x_\alpha) v_\alpha h)_{\alpha \in A} \rangle_K \\ &= \sum_{\alpha \in A} \langle v_\alpha k, \pi_\alpha(x_\alpha) v_\alpha h \rangle_{K_\alpha} = \sum_{\alpha \in A} \langle h, v_\alpha^* \pi_\alpha(x_\alpha) v_\alpha h \rangle_H \\ &= \sum_{\alpha \in A} \langle k, \varphi_\alpha(x_\alpha) h \rangle_H = \left\langle k, \sum_{\alpha \in A} \varphi_\alpha(x_\alpha) h \right\rangle \\ &= \langle k, \varphi(x) h \rangle, \end{aligned}$$

where the penultimate equality is because  $\sum_{\alpha} \varphi_\alpha(x_\alpha)$  converges to  $\varphi(x)$  in the  $\sigma$ -weak operator topology, hence also in the weak operator topology. Since we can take  $h$  and  $k$  in  $H$  arbitrary, it follows that  $v^* \pi(x) v = \varphi(x)$ .

Finally, let  $e_\alpha : K_\alpha \rightarrow K$  denote the embedding and denote by  $G_\alpha$  its image in  $K$ . Let  $k = (k_\alpha)_{\alpha \in A}$  be an element of  $K$ . Then for each  $\alpha \in A$  it follows that  $k_\alpha \in K_\alpha = \overline{\pi_\alpha[M_\alpha]v_\alpha H}$ .

Let  $\alpha \in A$ ,  $x \in M_\alpha$  and  $h \in H$ . Then  $\iota_\alpha(x)_\beta = \delta_{\alpha\beta}x$  for each  $\beta \in A$ , hence

$$(\pi(\iota_\alpha(x))v(h))_\beta = \pi_\beta(\iota_\alpha(x)_\beta)(v(h))_\beta = \pi_\beta(\delta_{\alpha\beta}x)v_\beta h = \delta_{\alpha\beta}\pi_\alpha(x)v_\alpha h = (e_\alpha \pi_\alpha(x)v_\alpha h)_\beta,$$

so  $\pi(\iota_\alpha(x))v(h) = e_\alpha \pi_\alpha(x)v_\alpha(h)$ , where we recall that  $e_\alpha : K_\alpha \rightarrow K$  is the embedding, whose image is  $G_\alpha \subseteq K$ . As a consequence, we have for each  $\alpha \in A$ :

$$G_\alpha = e_\alpha[K_\alpha] = e_\alpha[\overline{\pi_\alpha[M_\alpha]v_\alpha H}] \subseteq \overline{e_\alpha[\pi_\alpha[M_\alpha]v_\alpha H]} = \overline{\pi[\iota_\alpha[M_\alpha]]vH} \subseteq \overline{\pi[M]vH},$$

where we used that  $e_\alpha$  is bounded, so continuous in the first inclusion. Since clearly  $\bigvee_{\alpha \in A} G_\alpha = K$ , it follows that  $K \subseteq \overline{\pi[M]vH}$ , which implies  $K = \overline{\pi[M]vH}$ . We conclude that  $\pi$  is the minimal Stinespring representation corresponding to  $\varphi$ .  $\square$

**Lemma 80.** *Let  $M$  be a von Neumann algebra, and let  $(N_\alpha)_{\alpha \in A}$  be a collection of von Neumann algebras. Let  $N = \prod_{\alpha \in A} N_\alpha$ , and let  $\pi_\alpha : N \rightarrow N_\alpha$  be the projection on the  $\alpha$ -th coordinate. Then  $\iota : \mathbf{vN}(M, N) \rightarrow \prod_{\alpha \in A} \mathbf{vN}(M, N_\alpha)$ ,  $\varphi \mapsto \pi_\alpha \circ \varphi$  is an order isomorphism.*

*Proof.* Firstly,  $\pi_\alpha$  is a unital  $*$ -homomorphism, so certainly completely positive and subunital. Let  $H_\alpha$  be the Hilbert space such that  $N_\alpha$  is a von Neumann algebra on  $H_\alpha$ , hence  $N$  is a von Neumann algebra on  $H \stackrel{\text{def}}{=} \bigoplus_{\alpha \in A} H_\alpha$ . Let  $x = (x_\alpha)_{\alpha \in A}$  be an element of  $N$ . Then  $x$  is positive if and only if for each  $h = (h_\alpha)_{\alpha \in A}$  in  $H$  we have  $0 \leq \langle h, xh \rangle$ , i.e., if and only if  $0 \leq \sum_{\alpha \in A} \langle h_\alpha, x_\alpha h_\alpha \rangle$  from which it is clear that the positivity of each  $x_\alpha$  is sufficient for  $x$  to be positive. Since for fixed  $\beta \in A$ , we can choose  $h$  in such a way that  $h_\alpha = 0$  for each  $\alpha \neq \beta$ , it follows that  $x$  positive implies that  $\langle h_\beta, x_\beta h_\beta \rangle \geq 0$ , so it is also necessary for  $x$  to be positive that each  $x_\beta$  is positive. Thus  $x$  is positive if and only if each  $x_\alpha$  is positive. As a consequence, if  $(x^d)_{d \in D}$  is a monotonically ascending net in  $N_{\text{sa}}$  with supremum  $x$  and we write  $x^d = (x_\alpha^d)_{\alpha \in A}$  and  $x = (x_\alpha)_{\alpha \in A}$ , we have  $x_\alpha = \sup_{d \in D} x_\alpha^d$ , whence  $\pi_\alpha(x) = \sup_{d \in D} \pi_\alpha(x^d)$ , so  $\pi_\alpha$  is normal.

As a consequence,  $\iota$  is well defined. Moreover, we have  $\varphi \leq \psi$  in  $\mathbf{vN}(M, N)$  if and only if  $\psi - \varphi$  is positive, if and only if  $(\psi - \varphi)(x)$  is positive for each positive  $x \in M$  if and only if  $\pi_\alpha \circ (\psi - \varphi)(x)$  is positive for each  $x \in M$  and each  $\alpha \in A$  if and only if  $(\iota(\psi) - \iota(\varphi))(x)$  is positive for each positive  $x \in M$  if and only if  $\iota(\psi) - \iota(\varphi)$  is positive if and only if  $\iota(\varphi) \leq \iota(\psi)$ . So  $\iota$  is an order embedding. It is surjective, since  $\psi \in \prod_{\alpha \in A} \mathbf{vN}(M, N_\alpha)$ , then  $\psi = (\psi_\alpha)_{\alpha \in A}$  for some completely positive maps  $\psi_\alpha : M \rightarrow N_\alpha$ , and by the properties of the categorical product, that implies the existence of some  $\varphi = \langle \psi_\alpha \rangle_{\alpha \in A} : M \rightarrow N$  such that  $\pi_\alpha(\varphi) = \psi_\alpha$  for each  $\alpha \in A$ , i.e.,  $\iota(\varphi) = \psi$ .  $\square$

*Proof of Theorem 55.* Let  $M$  and  $N$  be hereditarily atomic von Neumann algebras. We show that the pointed dcpo  $\mathbf{vN}(M, N)$  is continuous. We first assume that  $N = B(H)$  for some finite-dimensional Hilbert space  $H$ . Since  $M$  is hereditarily atomic, we can write  $M = \prod_{\alpha \in A} M_\alpha$ , where  $M_\alpha$  is a matrix algebra. Let  $\varphi \in \mathbf{vN}(M, N)$ . It follows from combining Lemmas 78 and 79 that the minimal Stinespring representation  $(\pi, v, K)$  of  $\varphi$  can be obtained from the minimal Stinespring representations  $(\pi_\alpha, v_\alpha, K_\alpha)$  of  $\varphi_\alpha := \varphi \circ \iota_\alpha$ , and that  $\pi[M]'$  embeds into  $\prod_{\alpha \in A} B(K_\alpha)$ . Since  $(\pi_\alpha, v_\alpha, K_\alpha)$  is minimal,  $\pi_\alpha[M_\alpha]v_\alpha H$  is dense in  $K_\alpha$ . Since both  $M_\alpha$  and  $H$  are finite-dimensional, it follows that  $K_\alpha$  is finite-dimensional, too, so  $\prod_{\alpha \in A} B(K_\alpha)$  is hereditarily atomic. Since  $\pi[M]'$  embeds in this algebra, it is a hereditarily atomic von Neumann algebra, too, hence its unit interval  $[0, 1]_{\pi[M]'}$  is a continuous dcpo by Lemma 76. It now follows from Proposition 77 that  $\downarrow \varphi$  is a continuous dcpo. Thus all principal downsets in  $\mathbf{vN}(M, N)$  are continuous, so  $\mathbf{vN}(M, N)$  is continuous by [17, Proposition 2.2.17].

If  $N$  is an arbitrary hereditarily atomic von Neumann algebra, then  $N = \prod_{\beta \in B} B(H_\beta)$  for some finite-dimensional Hilbert spaces  $H_\beta$ . By Lemma 80 we have an order isomorphism  $\mathbf{vN}(M, N) \cong \prod_{\beta \in B} \mathbf{vN}(M, B(H_\beta))$ , and since the product of pointed continuous dcpos is continuous [18, Exercise I-2.18], it follows that  $\mathbf{vN}(M, N)$  is continuous. Hence  $\mathbf{vN}$  is enriched over continuous dcpos, so its dual  $\mathbf{Q}$  also is enriched over continuous dcpos.  $\square$

We proceed by showing that  $\mathbf{vN}$  is enriched over Kegelspitzen, whence  $\mathbf{HA}$  and  $\mathbf{Q}$  are also enriched over Kegelspitzen.

**Definition 81.** An *abstract convex set* or *barycentric algebra* is a set  $A$  equipped with for each  $r \in [0, 1]$  a binary operation  $+_r : A \times A \rightarrow A$  such that the following equational laws hold:

$$\begin{aligned} a +_1 b &= a; \\ a +_r a &= a; \\ a +_r b &= b +_{1-r} a; \\ (a +_p b) +_r c &= a +_{pr} \left( b +_{\frac{r-p}{1-pr}} c \right), \quad [r, p < 1]. \end{aligned}$$

If  $A$  has a distinguished element, which we usually denote by  $0$ , we call  $A$  *pointed*, in which case we can define scalar multiplication  $\cdot : [0, 1] \times A \rightarrow A$  by

$$r \cdot a := a +_r 0$$

for each  $r \in [0, 1]$  and each  $a \in A$ . For simplicity, we will sometimes write  $ra$  instead of  $r \cdot a$ . A map  $f : A \rightarrow B$  between barycentric algebras is called *affine* if  $f(a +_r b) = f(a) +_r f(b)$  for each  $a, b \in A$  and each  $r \in [0, 1]$ . If, in addition,  $A$  and  $B$  are pointed and  $f(0) = 0$ , we call  $f$  *linear*.

**Lemma 82.** *Let  $M$  and  $N$  be von Neumann algebras. Then  $\mathbf{vN}(M, N)$  is a barycentric algebra if we define  $\varphi +_r \psi := r\varphi + (1-r)\psi$  for each  $\varphi, \psi \in \mathbf{vN}(M, N)$  and each  $r \in [0, 1]$ .*

*Proof.* Clearly  $\varphi +_r \psi$  is normal. We show that it is completely positive. First assume that  $R$  is a von Neumann algebra and  $x, y \in R$  are positive. By [37, Theorem 4.2.2]  $x + y$  is positive, and  $rx$  is positive for each  $r \in [0, \infty)$ . Hence if  $r \in [0, 1]$  it follows that  $rx + (1-r)y$  is a positive element of  $R$ . Now let  $\varphi, \psi : M \rightarrow N$  be positive maps. So  $\varphi(x)$  and  $\psi(x)$  are positive for each positive  $x \in M$ , hence for  $r \in [0, 1]$ , we also have that  $(r\varphi + (1-r)\psi)(x) = r\varphi(x) + (1-r)\psi(x)$  is positive, so  $r\varphi + (1-r)\psi$  is a positive map.

Let  $n \in \mathbb{N}$ . Then for each  $r, s \in \mathbb{C}$  and each  $[x_{ij}]_{i,j=1}^n$  in  $M_n(M)$ , we have

$$\begin{aligned} M_n(r\varphi + s\psi)([x_{ij}]_{i,j=1}^n) &= [(r\varphi + s\psi)(x_{ij})]_{i,j=1}^n = [r\varphi(x_{ij}) + s\psi(x_{ij})]_{i,j=1}^n \\ &= r[\varphi(x_{ij})]_{i,j=1}^n + s[\psi(x_{ij})]_{i,j=1}^n = rM_n(\varphi)([x_{ij}]_{i,j=1}^n) + sM_n(\psi)([x_{ij}]_{i,j=1}^n) \\ &= (rM_n(\varphi) + sM_n(\psi))([x_{ij}]_{i,j=1}^n), \end{aligned}$$

hence  $M_n(r\varphi + s\psi) = rM_n(\varphi) + sM_n(\psi)$ . Assume that  $\varphi, \psi : M \rightarrow N$  are normal completely positive subunital maps. Then  $M_n(\varphi)$  and  $M_n(\psi)$  are positive, hence  $rM_n(\varphi) + (1-r)M_n(\psi)$  is positive, which equals  $M_n(r\varphi + (1-r)\psi)$ , and since  $n \in \mathbb{N}$  is arbitrary, we conclude that  $r\varphi + (1-r)\psi$  is completely positive. Moreover, since both  $\varphi$  and  $\psi$  are subunital, we have

$$(r\varphi + (1-r)\psi)(1_M) = r\varphi(1_M) + (1-r)\psi(1_M) \leq r1_N + (1-r)1_N = 1_N,$$

so  $r\varphi + (1-r)\psi$  is subunital. Finally,  $\varphi$  and  $\psi$  are normal, so continuous with respect to the  $\sigma$ -weak operator topology, hence so is their convex combination  $r\varphi + (1-r)\psi$ . Thus  $+_r$  defined by  $\varphi +_r \psi := r\varphi + (1-r)\psi$  for each  $r \in [0, 1]$  is a well-defined binary operation on  $\mathbf{vN}(M, N)$ . We now have  $\varphi +_1 \psi = 1\varphi + (1-1)\psi = \varphi$ ,  $\varphi +_r \varphi = r\varphi + (1-r)\varphi = \varphi$ , and  $\varphi +_r \psi = r\varphi + (1-r)\psi = (1-r)\psi + (1-(1-r))\varphi = \psi +_{1-r} \varphi$ . Let  $\omega \in \mathbf{vN}(M, N)$  and  $p, r \in [0, 1]$ . Then

$$\begin{aligned} (\varphi +_p \psi) +_r \omega &= r(\varphi +_p \psi) + (1-r)\omega = r(p\varphi + (1-p)\psi) + (1-r)\omega \\ &= rp\varphi + r(1-p)\psi + (1-r)\omega = rp\varphi + (1-rp) \left( \frac{r-rp}{1-rp}\psi + \frac{1-r}{1-rp}\omega \right) \\ &= \varphi +_{rp} \left( \frac{r-rp}{1-rp}\psi + \frac{1-r}{1-rp}\omega \right) = \varphi +_{rp} \left( \frac{r-rp}{1-rp}\psi + \frac{1-rp-(r-rp)}{1-rp}\omega \right) \\ &= \varphi +_{rp} \left( \frac{r-rp}{1-rp}\psi + \left(1 - \frac{r-rp}{1-rp}\right)\omega \right) = \varphi_{rp} + \left( \psi +_{\frac{r-rp}{1-rp}} \omega \right). \end{aligned}$$

We conclude that  $\mathbf{vN}(M, N)$  is indeed a barycentric algebra.  $\square$

Given von Neumann algebras  $M$  and  $N$ , the map  $0 : M \rightarrow N$  defined  $x \mapsto 0$  is normal, completely positive and subunital, hence we can take it as the distinguished element of  $\mathbf{vN}(M, N)$ .

**Lemma 83.** *Let  $M, N$  and  $R$  be von Neumann algebras and let  $\varphi : M \rightarrow N$  be a normal completely positive subunital map. Then*

$$\mathbf{vN}(R, \varphi) : \mathbf{vN}(R, M) \rightarrow \mathbf{vN}(R, N), \quad \psi \mapsto \varphi \circ \psi$$

and

$$\mathbf{vN}(\varphi, R) : \mathbf{vN}(N, R) \rightarrow \mathbf{vN}(M, R), \quad \psi \mapsto \psi \circ \varphi$$

are linear maps between barycentric algebras.

*Proof.* Since  $\varphi$  is linear in the sense of linear algebra, it follows that  $\mathbf{vN}(R, \varphi)$  is affine. Since  $\varphi(0) = 0$ , it follows that  $\mathbf{vN}(R, \varphi)$  is linear.

Write  $f = \mathbf{vN}(\varphi, R)$ . Then for  $\psi, \omega \in \mathbf{vN}(N, R)$ , we have

$$f(\psi +_r \omega) = (\psi +_r \omega) \circ \varphi,$$

hence for each  $x \in M$  we have

$$\begin{aligned} f(\psi +_r \omega)(x) &= ((\psi +_r \omega) \circ \varphi)(x) = ((r\psi + (1-r)\omega) \circ \varphi)(x) = (r\psi + (1-r)\omega)(\varphi(x)) \\ &= r\psi(\varphi(x)) + (1-r)\omega(\varphi(x)) = (r\psi \circ \varphi)(x) + ((1-r)\omega \circ \varphi)(x) \\ &= (r\psi \circ \varphi + (1-r)\omega \circ \varphi)(x) = (\psi \circ \varphi +_r \omega \circ \varphi)(x) = (f(\psi) +_r f(\omega))(x) \end{aligned}$$



hence  $f(\psi +_r \omega) = f(\psi) +_r f(\omega)$ , so  $f$  is linear. Furthermore, we have for each  $x \in M$ :  $f(0)(x) = (0 \circ \omega)(x) = 0(\omega(x)) = 0$ , so  $f(0) = 0$ , expressing that  $f$  is linear.  $\square$

**Definition 84.** An *ordered barycentric algebra* is a barycentric algebra  $A$  equipped with an order  $\leq$  such that for each  $r \in [0, 1]$  the operation  $+_r$  is monotone, i.e., for each  $a, b, c \in A$  such that  $a \leq b$  we have  $a +_r c \leq b +_r c$ .

**Lemma 85.** Let  $M$  and  $N$  be von Neumann algebras. Then  $\mathbf{vN}(M, N)$  is an ordered barycentric algebra under the usual order  $\varphi \leq \psi$  if and only if  $\psi - \varphi$  is positive.

*Proof.* Let  $x, y, z$  be positive elements on  $N$  such that  $x \leq y$ . Then  $y - x$  is positive, and since  $y - x = y + z - (x + z)$  is positive, it follows that  $x + z \leq y + z$ .

Let  $\varphi, \psi, \omega \in \mathbf{vN}(M, N)$  and assume that  $\varphi \leq \omega$ . Let  $x \in M$  be positive. Then  $\varphi(x) \leq \omega(x)$ , hence  $r\varphi(x) \leq r\omega(x)$ , so  $r\varphi(x) + (1-r)\psi(x) \leq r\omega(x) + (1-r)\psi(x)$ , hence  $\varphi +_r \psi \leq \omega +_r \psi$ .  $\square$

**Definition 86.** Let  $K$  be a pointed ordered barycentric algebra that is a dcpo in its underlying poset. Then we call  $K$  a *Kegelspitze* if for each  $r \in [0, 1]$  the map  $+_r : K \times K \rightarrow K$ , and scalar multiplication  $(r, a) \mapsto r \cdot a, [0, 1] \times K \rightarrow K$  are Scott continuous in both arguments. We denote the category of Kegelspitzen and Scott continuous linear maps by  $\mathbf{KS}$ .

**Proposition 87.** Let  $M$  and  $N$  be von Neumann algebras. Then  $\mathbf{vN}(M, N)$  is a Kegelspitze.

*Proof.* Upon inspecting the proof of [41, Proposition 5.2] that shows that  $\mathbf{vN}(M, N)$  is a pointed dcpo, the supremum  $\varphi$  of any directed set  $(\varphi_\alpha)_{\alpha \in A}$  in  $\mathbf{vN}(M, N)$  is calculated pointwise:  $\varphi : M \rightarrow N$  is the normal completely positive subunital map such that  $\varphi(x) = \bigvee_{\alpha \in A} \varphi_\alpha(x)$  for each  $x \in M$ , where the supremum of the  $\varphi_\alpha(x)$  is calculated in  $N_{\text{sa}}$ , the  $\mathbf{R}$ -vector space of all self-adjoint elements in  $N$  and is the limit of the  $\varphi_\alpha(x)$  with respect to the strong operator topology on  $N$  [37, Lemma 5.1.4]. This means that if  $H$  is a Hilbert space such that  $N$  is a von Neumann algebra on  $B(H)$ , we have that  $\|\varphi_\alpha(x)h - \varphi(x)h\|$  converges to 0 for each  $h \in H$ . Let  $r \in [0, 1]$  and fix  $\psi \in \mathbf{vN}(M, N)$ . Then for each  $h \in H$ , we have

$$\|(r\varphi_\alpha + (1-r)\psi)(x)h - (r\varphi + (1-r)\psi)(x)h\| = \|r\varphi_\alpha(x)h - r\varphi(x)h\| = |r|\|\varphi_\alpha(x)h - \varphi(x)h\|,$$

which clearly converges to 0, hence  $\bigvee_{\alpha \in A} (r\varphi_\alpha(x) + (1-r)\psi(x)) = r\varphi(x) + (1-r)\psi(x)$ , whence  $\bigvee_{\alpha \in A} (\varphi_\alpha +_r \psi) = \varphi +_r \psi$ . We conclude that  $+_r : \mathbf{vN}(M, N) \times \mathbf{vN}(M, N) \rightarrow \mathbf{vN}(M, N)$  is Scott continuous in the first variable, and since  $\varphi +_r \psi = \psi +_{1-r} \varphi$ , it also follows that  $+_r$  is Scott continuous in the second variable, hence Scott continuous overall.

Since  $r \cdot \varphi = \varphi +_r 0$ , it follows that scalar multiplication is Scott continuous in  $\varphi$ , so we only have to check that it is Scott continuous in  $r$ . So fix  $\varphi \in \mathbf{vN}(M, N)$ , and let  $D \subseteq [0, 1]$  be a directed set with supremum  $s$ . We need to show that  $\bigvee_{d \in D} d\varphi = s\varphi$ , hence for each  $x \in M$ , we need to show that  $(d\varphi(x))_{d \in D}$  converges to  $s\varphi(x)$  in the strong operator topology. Thus we need to show for each  $x \in M$  and each  $h \in H$  that  $\|d\varphi(x)h - s\varphi(x)h\|$  converges to zero. But  $\|d\varphi(x) - s\varphi(x)\| = |d - s|\|\varphi(x)\|$  and obviously  $|d - s|$  converges to 0 since  $s$  is the supremum of  $D$  in  $[0, 1]$ , which shows that scalar multiplication is also Scott continuous in both variables, hence Scott continuous overall.  $\square$

**Theorem 88.** The category  $\mathbf{vN}$  is enriched over  $\mathbf{KS}$ .

*Proof.* By Proposition 87, any homset in  $\mathbf{vN}$  is a Kegelspitze. We have to verify that for any von Neumann algebras  $M, N$ , and  $R$ , and any normal completely positive subunital map  $\varphi : M \rightarrow N$ , the maps

$$\mathbf{vN}(R, \varphi) : \mathbf{vN}(R, M) \rightarrow \mathbf{vN}(R, N), \quad \psi \mapsto \varphi \circ \psi$$

and

$$\mathbf{vN}(\varphi, R) : \mathbf{vN}(N, R) \rightarrow \mathbf{vN}(M, R), \quad \psi \mapsto \psi \circ \varphi$$

are morphisms in  $\mathbf{KS}$ . It follows from [41, Theorem 5.3] that  $\mathbf{vN}$  is enriched over  $\mathbf{DCPO}_{\perp 1}$ , hence the morphisms above are Scott continuous. By Lemma 83, the maps are linear, hence indeed morphisms in  $\mathbf{KS}$ .  $\square$

APPENDIX E  
THE ISOMORPHISM  $r_{\mathcal{O}}$

**Definition 89.** Let  $\mathcal{L} : \text{Set} \rightarrow \text{DCPO}$  be the functor defined by

$$\begin{aligned}\mathcal{L}X &\stackrel{\text{def}}{=} (X, \sqsubseteq), \text{ where } \sqsubseteq \text{ is the discrete order on } X \\ \mathcal{L}f &\stackrel{\text{def}}{=} f\end{aligned}$$

**Definition 90.** A complex Banach algebra  $A$  is called a *\*-algebra* if it equipped with an idempotent conjugate-linear map  $*$  :  $A \rightarrow A$  such that  $(ab)^* = b^*a^*$  for each  $a, b \in A$ . If, in addition,  $\|a^*a\| = \|a\|^2$  for each  $a \in A$ , we call  $A$  a *C\*-algebra*.

In particular any von Neumann algebra is a C\*-algebra. More generally, for each Hilbert space  $H$ , every norm-closed \*-subalgebra of  $B(H)$  is a C\*-algebra. The converse holds as well: if  $A$  is a C\*-algebra, any \*-homomorphism  $\pi : A \rightarrow B(H)$  for some Hilbert space  $H$  is called a *representation*. The Gelfand-Naimark Representation Theorem [35, Corollary II.6.4.10] states that any C\*-algebra has a faithful representation.

**Theorem 91.** Any C\*-algebra that is \*-isomorphic to a von Neumann algebra is called a W\*-algebra.

Let  $V$  be a Banach space. Its dual space, i.e., the space of all continuous linear maps  $\varphi : V \rightarrow \mathbb{C}$ , is denoted by  $V^*$ . Note that we can always isometrically embed  $V$  into its double dual  $V^{**}$  via the map  $v \mapsto \hat{v}$ , where  $\hat{v} : V^* \rightarrow \mathbb{C}$  is given by  $\hat{v}(\varphi) = \varphi(v)$  for each  $\varphi \in V^*$ .

The following theorem gives an alternative characterization of W\*-algebras:

**Theorem 92.** [35, III.2.1.8, Theorems III.2.4.1 & III.2.4.2] Let  $A$  be a C\*-algebra. Then  $A$  is a W\*-algebra if and only if there is a Banach space  $F$  such that  $A$  is isometrically isomorphic to the dual  $F^*$  of  $F$ . Under the isometric embedding of  $F$  into  $F^{**}$ , and by the isometric isomorphism  $A^* \cong F^{**}$ ,  $F$  is isometrically isomorphic to  $A_*$ , the subspace of  $A^*$  consisting of all normal functionals on  $A$ .

Hence if  $M$  is a W\*-algebra, then  $M_*$  is up to isometric isomorphism is the unique Banach space whose dual is isometrically isomorphic to  $M$ . We call  $M_*$  as the *predual* of  $M$ .

Let  $X$  be a set. From Example 49 we already know that  $\ell^\infty(X)$  is a von Neumann algebra. Its predual is the the Banach space of all functions  $f : X \rightarrow \mathbb{C}$  such that  $\sum_{x \in X} |f(x)| < \infty$ , which we denote by  $\ell^1(X)$ :

**Lemma 93.** Let  $X$  be a set. The predual of  $\ell^\infty(X)$  is  $\ell^1(X)$ , where the isometric isomorphism  $\ell^\infty(X) \rightarrow \ell^1(X)^*$  is given by  $f \mapsto \tilde{f}$  with  $\tilde{f} : \ell^1(X) \rightarrow \mathbb{C}$  given by  $\tilde{f}(g) = \sum_{x \in X} f(x)g(x)$  for each  $g \in \ell^1(X)$ .

*Proof.* The isomorphism  $\ell^\infty(X) \rightarrow \ell^1(X)^*$  is asserted in [38, Table B.1, p. 547] and proven on [38, p. 551]. □

Let  $\mathcal{D}(X)$  be the set of all  $g \in \ell^1(X)$  such that  $g(x) \geq 0$  for each  $x \in X$  and  $\sum_{x \in X} g(x) \leq 1$ , which we order by  $g \leq g'$  if and only if  $g(x) \leq g'(x)$  for each  $x \in X$ .

**Lemma 94.** We have an isometric isomorphism  $\zeta : \ell^1(X) \rightarrow \ell^\infty(X)_*$  defined by

$$\zeta(g)(f) = \sum_{x \in X} f(x)g(x)$$

for each  $g \in \ell^1(X)$  and each  $f \in \ell^\infty(X)$ .

*Proof.* If  $M$  is a C\*-algebra, then any positive map  $\omega : M \rightarrow \mathbb{C}$  is completely positive [46, Theorem 1.2.4]. Hence  $\text{vN}(M, \mathbb{C})$  is the space of all normal positive subunital linear maps, hence a subspace of  $M_*$ . Let  $X$  be a set. Recall Example 49 which states that  $\ell^\infty(X)$  is a von Neumann algebra on  $\ell^2(X)$  by acting on the latter by left multiplication. As a consequence,  $f \in \ell^\infty(X)$  is positive if and only if for each  $h \in \ell^2(X)$ , we have  $\langle h, fh \rangle \geq 0$ . Since  $\langle h, fh \rangle = \sum_{x \in X} \overline{h(x)}f(x)h(x) = \sum_{x \in X} f(x)|h(x)|^2$ , and since for each  $y \in X$ , the function  $e_y : X \rightarrow \mathbb{C}$  defined by  $e_y(x) = \delta_{x,y}$  for each  $x \in X$  is an element of  $\ell^2(X)$ , it follows that  $f$  is positive if and only if  $f(x) \geq 0$  for each  $x \in X$ .

Given a continuous linear map  $f : V \rightarrow W$  between Banach spaces  $V$  and  $W$ , we have an induced continuous linear map  $f^* : W^* \rightarrow V^*$ ,  $\varphi \mapsto \varphi \circ f$ . If  $f$  is an isometric isomorphism, so is  $f^*$  since we have

$$\|f^*(\varphi)\| = \|\varphi \circ f\| = \sup\{|\varphi \circ f(v)| : v \in V, \|v\| = 1\} = \sup\{|\varphi(w)| : w \in W, \|w\| = 1\} = \|\varphi\|,$$

where we used that  $f$  is an isometric isomorphism in the penultimate equality. Hence  $f^*$  is an isometry, hence in particular injective. Moreover, if  $\psi \in V^*$ , then  $\varphi \stackrel{\text{def}}{=} \psi \circ f^{-1} = (f^{-1})^*(\psi) \in W^*$ , hence  $f^*(\varphi) = \varphi \circ f = \psi$ , so  $f^*$  is surjective.

Let  $X$  be a set and let  $M = \ell^\infty(X)$ . Denote the isometric isomorphism from Lemma 93 by  $\theta$ . Then we have an isometric isomorphism  $\theta^* : \ell^1(X)^{**} \rightarrow \ell^\infty(X)^*$ . If we denote the canonical embedding  $\ell^1(X) \rightarrow \ell^1(X)^{**}$  by  $\eta$ , then we have an isometric embedding  $\zeta \stackrel{\text{def}}{=} \theta^* \circ \eta : \ell^1(X) \rightarrow \ell^\infty(X)^*$  whose image is  $\ell^\infty(X)_*$ . For each  $g \in \ell^1(X)$  and  $f \in \ell^\infty(X)$ , we have

$$\zeta(g)(f) = ((\theta^* \circ \eta)(g))(f) = (\theta^*(\hat{g}))(f) = (\hat{g} \circ \theta)(f) = \hat{g}(\tilde{f}) = \tilde{f}(g) = \sum_{x \in X} f(x)g(x).$$

□

**Proposition 95.** *The isometric isomorphism  $\zeta : \ell^1(X) \rightarrow \ell^\infty(X)_*$  restricts to an isomorphism of Kegelspitzen  $\mathcal{D}(X) \rightarrow \mathbf{vN}(\ell^\infty(X), \mathbb{C})$ .*

*Proof.* Let  $g \in \ell^1(X)$ . Then  $\zeta(g) \in \ell^\infty(X)^*$  is positive if and only if  $\zeta(g)(f)$  is positive for each positive  $f \in \ell^\infty(X)$ , i.e.,  $\sum_{x \in X} f(x)g(x) \geq 0$  for each positive  $f \in \ell^\infty(X)$ . Since  $f$  is positive if and only if  $f(x) \geq 0$  for each  $x \in X$ , and for each  $y \in X$  the function  $e_y : X \rightarrow \mathbb{C}$  defined by  $e_y(x) = \delta_{x,y}$  for each  $x \in X$  is an element of  $\ell^\infty(X)$ , it follows that  $\zeta(g)(f)$  is positive if and only if  $g(x) \geq 0$  for each  $x \in X$ . It also follows that if  $g' \in \ell^1(X)$  is another element such that  $\zeta(g')$  is positive, then  $\zeta(g) \leq \zeta(g')$  if and only if  $\zeta(g' - g)$  is positive if and only if  $g(x) \leq g'(x)$  for each  $x \in X$ .

Let  $\omega \in \ell^\infty(X)^*$  be positive. By [35, Proposition II.6.2.5],  $\omega$  is bounded with norm  $\|\omega\| = \omega(1)$ . Hence,  $\omega$  is subunital if and only if  $\|\omega\| \leq 1$ . Hence if  $\zeta(g)$  positive, it is bounded with norm  $\|\zeta(g)\| = \zeta(g)(1)$ , so  $\|\zeta(g)\| = \sum_{x \in X} g(x)$ . Thus  $\zeta(g)$  is positive and subunital if and only if  $\|g\| = \|\zeta(g)\| = \sum_{x \in X} g(x) \leq 1$ , where we used that  $\zeta$  is an isometry. It follows that  $\zeta$  restricts to an isometric isomorphism between  $\mathbf{vN}(\ell^\infty(X), \mathbb{C})$  and  $\mathcal{D}(X)$ , which is also an order isomorphism. Since isometric isomorphisms in particular preserve the convex structure, it follows the restriction of  $\zeta$  is a Kegelspitze isomorphism. □

**Proposition 96.** *For any set  $X$ , there exists an isomorphism of Kegelspitzen  $r_X : \mathbf{Q}(\ell^\infty(1), \ell^\infty(X)) \cong \mathbf{DCPO}_{\mathcal{M}}(1, \mathcal{L}X) : r_X^{-1}$ . Moreover,  $r_X$  restricts to an isomorphism of sets  $r_X : \mathbf{Q}_*(\ell^\infty(1), \ell^\infty(X)) \cong \mathbf{TD}(1, \mathcal{L}X) : r_X^{-1}$ .*

*Proof.* Firstly, we have an isomorphism

$$\alpha : \mathbf{DCPO}_{\mathcal{M}}(1, \mathcal{L}X) = \mathbf{DCPO}(1, \mathcal{M}\mathcal{L}X) \rightarrow \mathcal{M}\mathcal{L}(X) = \mathcal{D}(X), \quad f \mapsto f(1).$$

Furthermore, we have an isomorphism  $i : \mathbb{C} \rightarrow \ell^\infty(1)$  that to each  $\lambda \in \mathbb{C}$  assigns the function  $c_\lambda \in \ell^\infty(1)$  defined by  $c_\lambda(1) = \lambda$ . This isomorphism induces an isomorphism

$$\iota : \mathbf{vN}(\ell^\infty(X), \mathbb{C}) \rightarrow \mathbf{vN}(\ell^\infty(X), \ell^\infty(1)) = \mathbf{Q}(\ell^\infty(1), \ell^\infty(X)), \quad f \mapsto i \circ f.$$

It follows that  $r_X^{-1} : \mathbf{DCPO}_{\mathcal{M}}(1, \mathcal{L}X) \rightarrow \mathbf{Q}(\ell^\infty(1), \ell^\infty(X))$  given by  $\iota \circ \zeta \circ \alpha$  is the required isomorphism.

Note that  $\ell^\infty : \mathbf{Set} \rightarrow \mathbf{Q}_*$  is a functor if for each function  $f : X \rightarrow Y$  between sets, we define  $\ell^\infty(f) : \ell^\infty(Y) \rightarrow \ell^\infty(X)$  by  $g \mapsto g \circ f$ . It is easy to see that this functor is fully faithful: if  $f, f' : X \rightarrow Y$  such that  $\ell^\infty(f) = \ell^\infty(f')$ . For  $y \in Y$ , let  $e_y \in \ell^\infty(Y)$  be given by  $e_y(y') = \delta_{y,y'}$ . Then for each  $x \in X$ , we have

$$1 = e_{f(x)}(f(x)) = (e_{f(x)} \circ f)(x) = \ell^\infty(f)(e_{f(x)})(x) = \ell^\infty(f')(e_{f(x)})(x) = (e_{f(x)} \circ f')(x) = e_{f(x)}(f'(x)) = \delta_{f(x), f'(x)},$$

hence  $f(x) = f'(x)$  for each  $x \in X$ , so  $f = f'$ , showing that  $\ell^\infty$  is faithful. Let  $k \in \ell^\infty(X)$ . Then  $k = g \circ f$  for some  $g \in \ell^\infty(Y)$ . Indeed, define  $g$  as follows. If  $y \in Y \setminus f[X]$ , let  $g(y) = 0$ . If  $y \in f[X]$ , choose some  $x \in f^{-1}[\{y\}]$  and define  $g(y) = k(f(x))$ . Clearly this is independent of the choice of  $x \in f^{-1}[y]$ . Then  $\sup_{y \in Y} |g(y)| \leq \sup_{x \in X} |k(x)| < \infty$ , so  $g \in \ell^\infty(Y)$  and by construction we have  $k = g \circ f$ . So  $\ell^\infty$  is also full.

Now consider the following diagram: **TODO: insert diagram**

Since it consists only of isomorphisms and two injective functors (note that  $\mathcal{L}$  is fully faithful by construction), commutativity of the diagram implies that  $r_X$  restricts to an isomorphism  $\mathbf{Q}_*(\ell^\infty(1), \ell^\infty(X)) \rightarrow \mathbf{TD}(1, \mathcal{L}X)$ . In order to show that the diagram indeed is commutative, let  $f \in \mathbf{Set}(1, X)$ , want write  $f(1) = x$ . Then  $\ell^\infty(f) \in \mathbf{Q}_*(\ell^1(X), \ell^\infty(X)) = \mathbf{HA}_*(\ell^\infty(X), \ell^\infty(1))$  is given by  $\ell^\infty(f)(g) = g \circ f$ , i.e.,  $(\ell^\infty(f)(g))(1) = (g \circ f)(1) = g(x)$ , hence  $\ell^\infty(f)(g) = c_{g(x)}$ .

In the other direction, we have  $J(f)(1) = (\eta_{\mathcal{L}X} \circ f)(1) = \eta_{\mathcal{L}X}(x) = \delta_x$ , hence

$$(r_X^{-1} \circ J(f)) = (\iota \circ \zeta \circ \alpha)(J(f)) = (\iota \circ \zeta)(J(f)(1)) = (\iota \circ \zeta)(\delta_x),$$

which is a function  $\ell^\infty(X) \rightarrow \ell^\infty(1)$ , hence for  $g \in \ell^\infty(X)$ , we have

$$(r_X^{-1} \circ J(f))(g) = (\iota \circ \zeta)(\delta_x)(g) = \iota \left( \sum_{y \in X} \delta_x(y)g(y) \right) = \iota(g(x)) = i(g(x)) = c_{g(x)}.$$

Thus  $(r_X^{-1} \circ J(f))(g) = c_{g(x)} = \ell^\infty(f)(g)$ , so  $J(f) = r_X \circ \ell^\infty(f)$ , which shows that the diagram indeed commutes. □

APPENDIX F  
CONSTRUCTION OF THE LIFT ISOMORPHISM

First, we prove a more general proposition.

**Proposition 97.** *Given a dcpo  $X$ , HA-algebras  $A, B$ , and a discrete dcpo  $Y$ , there exists a Scott-continuous bijection*

$$\mathbf{DCPO}(X \times Y, \mathbf{Q}(A, B)) \cong \mathbf{DCPO}(X, \mathbf{Q}(\ell^\infty(\mathcal{R}Y) \otimes A, B)),$$

where  $\mathcal{R}Y$  is the underlying set of the dcpo  $Y$ .

*Proof.* This is established via the following sequence of Scott-continuous bijections:

$$\begin{array}{l} \frac{\mathbf{DCPO}(X \times Y, \mathbf{Q}(A, B))}{\mathbf{DCPO}(X, [Y \rightarrow \mathbf{Q}(A, B)])} \text{ (currying)} \\ \frac{\mathbf{DCPO}(X, [Y \rightarrow \mathbf{Q}(A, B)])}{\mathbf{DCPO}(X, \prod_{|Y|} \mathbf{Q}(A, B))} \text{ (} Y \text{ is discrete)} \\ \frac{\mathbf{DCPO}(X, \prod_{|Y|} \mathbf{Q}(A, B))}{\mathbf{DCPO}(X, \mathbf{Q}(\prod_{|Y|} A, B))} \text{ (} \mathbf{Q} \text{ has DCPO-enriched coproducts)} \\ \frac{\mathbf{DCPO}(X, \mathbf{Q}(\prod_{|Y|} A, B))}{\mathbf{DCPO}(X, \mathbf{Q}(\prod_{|Y|} (\mathbb{C} \otimes A), B))} \text{ (} \mathbb{C} \text{ is the monoidal unit)} \\ \frac{\mathbf{DCPO}(X, \mathbf{Q}(\prod_{|Y|} (\mathbb{C} \otimes A), B))}{\mathbf{DCPO}(X, \mathbf{Q}((\prod_{|Y|} \mathbb{C}) \otimes A, B))} \text{ (} (- \otimes A) \text{ preserves coproducts)} \\ \frac{\mathbf{DCPO}(X, \mathbf{Q}((\prod_{|Y|} \mathbb{C}) \otimes A, B))}{\mathbf{DCPO}(X, \mathbf{Q}(\ell^\infty(\mathcal{R}Y) \otimes A, B))} \text{ (} \ell^\infty(\mathcal{R}Y) \cong \prod_{|Y|} \mathbb{C} \text{)} \end{array}$$

where  $|Y|$  indicates the cardinality of  $Y$ . □

We can now show the proposition that is of primary interest to us.

*Proof of Proposition 61.* First, recall that if  $\mathbf{O}$  is a quantum observable type, then the classical observable type  $|\mathbf{O}|$  is interpreted as a discrete dcpo. Then:

$$\begin{array}{l} \frac{\mathbf{DCPO}([\Phi] \times [|\mathbf{O}|], \mathbf{Q}([\Gamma], [\mathbf{A}]))}{\mathbf{DCPO}([\Phi], \mathbf{Q}(\ell^\infty(\mathcal{R}[|\mathbf{O}|]) \otimes [\Gamma], [\mathbf{A}]))} \text{ (Proposition 97)} \\ \frac{\mathbf{DCPO}([\Phi], \mathbf{Q}(\ell^\infty(\mathcal{R}[|\mathbf{O}|]) \otimes [\Gamma], [\mathbf{A}]))}{\mathbf{DCPO}([\Phi], \mathbf{Q}([\mathbf{O}] \otimes [\Gamma], [\mathbf{A}]))} \text{ (} \ell^\infty(\mathcal{R}[|\mathbf{O}|]) \cong [|\mathbf{O}|] \text{)} \end{array}$$

□

APPENDIX G  
PROOF OF STRONG ADEQUACY

In this appendix we provide a proof of Theorem 65 for the classical fragment of the language. Adequacy for the whole language is work-in-progress. We begin by stating a corollary for the soundness theorem.

**Corollary 98.** *For any closed term  $\cdot \vdash M : A$ , we have:*

$$\llbracket M \rrbracket \geq \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

*Proof.* First, let us decompose the convex sum on the right-hand side.

$$\sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket = \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sum_{V \in F} P(M \rightarrow_* V) \llbracket V \rrbracket \quad (\text{Definition})$$

$$= \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sum_{V \in F} \left( \sup_{i \in \mathbb{N}} P(M \rightarrow_{\leq i} V) \right) \llbracket V \rrbracket \quad (\text{Definition})$$

$$= \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sup_{i \in \mathbb{N}} \sum_{V \in F} P(M \rightarrow_{\leq i} V) \llbracket V \rrbracket \quad \left( \text{Scott-continuity of } \sum_i r_i a_i \text{ in each } r_i \right).$$

Therefore, it suffices to show that

$$\llbracket M \rrbracket \geq \sum_{V \in F} P(M \rightarrow_{\leq i} V) \llbracket V \rrbracket \quad (7)$$

for any choice of finite  $F \subseteq \text{Val}(M)$  and  $i \in \mathbb{N}$ . This can now be shown by induction on  $i$ . If  $M \in F$  (which means  $M$  is a value), then (7) is a strict equality. Assume  $M \notin F$ . If  $i = 0$ , then the right-hand side of (7) is 0 and so the inequality holds. For the step case, if  $M$  is a value, then RHS is 0 and the inequality holds. Otherwise:

$$\begin{aligned} \sum_{V \in F} P(M \rightarrow_{\leq i+1} V) \llbracket V \rrbracket &= \sum_{V \in F} \sum_{M \xrightarrow{p} M'} p \cdot P(M' \rightarrow_{\leq i} V) \llbracket V \rrbracket \\ &= \sum_{M \xrightarrow{p} M'} p \cdot \sum_{V \in F} P(M' \rightarrow_{\leq i} V) \llbracket V \rrbracket \\ &\leq \sum_{M \xrightarrow{p} M'} p \cdot \llbracket M' \rrbracket \quad (\text{IH for } M') \\ &= \llbracket M \rrbracket \quad (\text{Soundness}) \end{aligned}$$

where we also implicitly used the fact that  $\text{Val}(M') \subseteq \text{Val}(M)$ . □

The remainder of the appendix is dedicated to showing the converse inequality, which is considerably more difficult to prove.

### A. Overview of the Proof Strategy

The proof of strong adequacy requires considerable effort. Our proof strategy consists in formulating logical relations that we use to prove our adequacy result. These logical relations are described in Theorem 138 and the design of our logical relations follows that of Claire Jones in her thesis [11]. Once this theorem is proved, the proof of adequacy is fairly straightforward. We use the logical relations to establish some useful closure properties in Subsection G-F and this allows us to easily prove Lemma 148, which is often called the Fundamental Lemma. This lemma easily implies Strong Adequacy as we show.

Most of the effort in proving our Strong Adequacy result lies in the proof of Theorem 138. It is not possible to use the properties (A1) – (A4) as a definition of the relations, because then condition (A4) would be defined via non-well-founded induction. The proof of the existence of this family of relations is not obvious. We use techniques from [15], [16] (which are in turn based on ideas from [9]) to show the existence of these relations. The main idea of the proof of existence is to define, for every type  $A$ , a category  $\mathbf{R}(A)$  of logical relations with a suitable notion of morphism. We then show that every such category has sufficient structure to construct parameterised initial algebras (Proposition 118). We may then define functors on these categories (Definition 125) which construct logical relations in the same manner as they are needed in Theorem 138. These functors are  $\omega$ -cocontinuous (Proposition 127) which means that we may form (parameterised) initial algebras using them. This allows us to define an *augmented interpretation of types* on the categories  $\mathbf{R}(A)$  which satisfies some important coherence conditions with respect to the standard interpretation of types (Corollary 136). These coherence conditions show that each augmented interpretation  $\llbracket A \rrbracket$  of a type  $A$  contains the standard interpretation  $\llbracket A \rrbracket$ , together with the logical relation that we need, as shown in Theorem 138.

## B. Logical Relations

**Assumption 99.** Throughout this appendix, we assume that all types are closed, unless otherwise noted.

**Definition 100.** For each type  $A$ , we write:

- $\text{Val}(A) \stackrel{\text{def}}{=} \{V \mid V \text{ is a value and } \cdot \vdash V : A\}$ .
- $\text{Prog}(A) \stackrel{\text{def}}{=} \{M \mid M \text{ is a term and } \cdot \vdash M : A\}$ .

Next, we define sets of relations that are parameterised by dcpo's  $X$  from our semantic category, types  $A$  from our language and partial deterministic embeddings  $e_X : X \rightarrow \llbracket A \rrbracket$  which show how  $X$  approximates  $\llbracket A \rrbracket$ . We shall write relation membership in infix notation, that is, for a binary relation  $\triangleleft$ , we write  $v \triangleleft V$  to indicate  $(v, V) \in \triangleleft$ .

**Definition 101.** For any dcpo  $X$ , type  $A$  and morphism  $e : X \rightarrow \llbracket A \rrbracket$  in  $\mathbf{PD}_e$ , let:

$$\text{ValRel}(X, A, e) = \{ \triangleleft_{X,A}^e \subseteq \mathbf{TD}(1, X) \times \text{Val}(A) \mid \forall V \in \text{Val}(A). (-) \triangleleft_{X,A}^e V \text{ is a Scott closed subset of } \mathbf{TD}(1, X) \text{ and } \forall V \in \text{Val}(A). v \triangleleft_{X,A}^e V \Rightarrow e \circ v \leq \llbracket V \rrbracket \}.$$

*Remark 102.* In the above definition, relations  $\triangleleft_{X,A}^e \in \text{ValRel}(X, A, e)$  can be seen as ternary relations  $\triangleleft_{X,A}^e \subseteq \mathbf{TD}(1, X) \times \text{Val}(A) \times \{e\}$ . However, since there is no choice for the third component, we prefer to see them as binary relations that are parameterised by the embeddings  $e$ . Indeed, this leads to a much nicer notation. We shall also sometimes indicate the parameters  $X, A$  and  $e$  of the relation in order to avoid confusion as to which set  $\text{ValRel}(X, A, e)$  it belongs to.

The relations we need for the adequacy proof inhabit the sets  $\text{ValRel}(\llbracket A \rrbracket, A, \text{id}_{\llbracket A \rrbracket})$ . In the remainder of the appendix, we will show how to choose exactly one relation (the one we need) from each of those sets.

Before we may define the relation constructors we need, we have to introduce some auxiliary definitions.

**Definition 103.** Let  $M : A$  and  $N : A$  be closed terms of the same type. We define

$$\text{Paths}(M, N) \stackrel{\text{def}}{=} \left\{ \pi \mid \pi = \left( M = M_0 \xrightarrow{p_0} M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \dots \xrightarrow{p_n} M_n = N \right) \text{ is a reduction path} \right\}.$$

In other words,  $\text{Paths}(M, N)$  is the set of all reduction paths from  $M$  to  $N$ . The *probability weight* of a path  $\pi \in \text{Paths}(M, N)$  is  $P(\pi) \stackrel{\text{def}}{=} \prod_{i=0}^n p_i$ , i.e., it is simply the product of all the probabilities of single-step reductions within the path. The *set of terminal reduction paths of  $M$*  is

$$\text{TPaths}(M) \stackrel{\text{def}}{=} \bigcup_{V \in \text{Val}(A)} \text{Paths}(M, V).$$

Thus the endpoint of any path  $\pi \in \text{TPaths}(M)$  is a value. If  $\pi \in \text{Paths}(M, W)$ , where  $W$  is a value, then we shall write  $V_\pi \stackrel{\text{def}}{=} W$ . That is, for a path  $\pi \in \text{TPaths}(M)$ , the notation  $V_\pi$  indicates the endpoint of the path  $\pi$  which is indeed a value.

*Remark 104.* We also note that for each closed term  $M$ , the set  $\text{TPaths}(M)$  is countable.

The next definition we introduce is crucial for the proof of strong adequacy.

**Definition 105.** Given a relation  $\triangleleft_{X,A}^e \in \text{ValRel}(X, A, e)$  and a term  $\cdot \vdash M : A$ , let  $\mathcal{S}(\triangleleft_{X,A}^e; M)$  be the Scott-closure in  $\mathbf{DCPO}_{\mathcal{M}}(1, X)$  of the set

$$\mathcal{S}_0(\triangleleft_{X,A}^e; M) \stackrel{\text{def}}{=} \left\{ \sum_{\pi \in F} P(\pi) v_\pi \mid F \subseteq \text{TPaths}(M), F \text{ is finite and } v_\pi \triangleleft_{X,A}^e V_\pi \text{ for each } \pi \in F \right\}. \quad (8)$$

In other words,  $\mathcal{S}(\triangleleft_{X,A}^e; M)$  is the smallest Scott-closed subset of  $\mathbf{DCPO}_{\mathcal{M}}(1, X)$  which contains all morphisms of the form in (8). For a subset  $U \subseteq \mathbf{DCPO}_{\mathcal{M}}(1, X)$ , we write  $\bar{U}$  to indicate its Scott-closure in  $\mathbf{DCPO}_{\mathcal{M}}(1, X)$ .

**Lemma 106.** For any value  $V$ , we have  $\mathcal{S}(\triangleleft_{X,A}^e; V) = \overline{\{v \mid v \triangleleft_{X,A}^e V\}} \cup \{0\} = \overline{\{v \mid v \triangleleft_{X,A}^e V\}} \cup \{0\}$ .

*Proof.* This is because all of the sums in (8) are singleton sums or the empty sum.  $\square$

**Lemma 107** ([11, Lemma 8.4]). Let  $Y$  be a dcpo and let  $\{X_i\}_{i \in F}$  be a finite collection of dcpo's. Let  $f : \prod_i X_i \rightarrow Y$  be a Scott-continuous function. Let  $C_Y$  be a Scott-closed subset of  $Y$ . Let  $U_i \subseteq X_i$  be arbitrary subsets, such that  $f(\prod_i U_i) \subseteq C_Y$ . Then  $f(\prod_i \bar{U}_i) \subseteq C_Y$ , where  $\bar{U}_i$  is the Scott-closure of  $U_i$  in  $X_i$ .

**Lemma 108.** Let  $\triangleleft_{X_1,A}^{e_1}$  and  $\triangleleft_{X_2,A}^{e_2}$  be two logical relations and  $\cdot \vdash M : A$  a term. Assume that  $g : X_1 \rightarrow X_2$  is a morphism, such that  $v \triangleleft_{X_1,A}^{e_1} V$  implies  $g \circ v \in \mathcal{S}(\triangleleft_{X_2,A}^{e_2}; V)$ , for any  $V \in \text{Val}(M)$ . If  $m \in \mathcal{S}(\triangleleft_{X_1,A}^{e_1}; M)$ , then  $g \circ m \in \mathcal{S}(\triangleleft_{X_2,A}^{e_2}; M)$ .

*Proof.* By Lemma 107, it suffices to show that

$$\left( g \circ \sum_{\pi \in F} P(\pi)v_\pi \right) \in \mathcal{S}(\triangleleft_{X_2, A}^{e_2}; M)$$

for any choice of finite  $F \subseteq \text{TPaths}(M)$  and morphisms  $v_\pi$  with  $v_\pi \triangleleft_{X_1, A}^{e_1} V_\pi$ . We have

$$g \circ \sum_{\pi \in F} P(\pi)v_\pi = \sum_{\pi \in F} P(\pi)(g \circ v_\pi),$$

where the equality follows by linearity of  $(g \circ -)$ . Next, for each  $v_\pi$ , by assumption  $g \circ v_\pi \in \mathcal{S}(\triangleleft_{X_2, A}^{e_2}; V_\pi)$ . Therefore by applying Lemma 106, it follows  $g \circ v_\pi \in \overline{\{v' \mid v' \triangleleft_{X_2, A}^{e_2} V_\pi\}} \cup \{0\}$ . Now, consider the function

$$\sum_{\pi \in F} P(\pi)(-): \prod_{|F|} \mathbf{DCPO}_{\mathcal{M}}(1, X_2) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, X_2).$$

This function is continuous, so by Lemma 107 again, it suffices to show that

$$\sum_{\pi \in F} P(\pi)m'_\pi = \sum_{\substack{\pi \in F \\ m'_\pi \neq 0}} P(\pi)m'_\pi \in \mathcal{S}(\triangleleft_{X_2, A}^{e_2}; M),$$

where either  $m'_\pi = 0$  or  $m'_\pi \triangleleft_{X_2, A}^{e_2} V_\pi$  for each  $\pi \in F$ . Since the summands where  $m'_\pi = 0$  do not affect the sum, it suffices to show that this is true under the assumption that  $m'_\pi \triangleleft_{X_2, A}^{e_2} V_\pi$ . But this is true by definition of  $\mathcal{S}(\triangleleft_{X_2, A}^{e_2}; M)$ .  $\square$

Next, we define important *closure relations* which we use for terms.

**Definition 109.** If  $\triangleleft_{X, A}^e \in \text{ValRel}(X, A, e)$ , let  $\overline{\triangleleft_{X, A}^e} \subseteq \mathbf{DCPO}_{\mathcal{M}}(1, X) \times \text{Prog}(A)$  be the relation defined by

$$m \overline{\triangleleft_{X, A}^e} M \text{ iff } m \in \mathcal{S}(\triangleleft_{X, A}^e; M).$$

**Lemma 110.** For any term  $\cdot \vdash M : A$  and  $\triangleleft_{X, A}^e \in \text{ValRel}(X, A, e)$ , the set  $(-) \overline{\triangleleft_{X, A}^e} M$  is a Scott-closed subset of  $\mathbf{DCPO}_{\mathcal{M}}(1, X)$ .

*Proof.* This follows immediately by definition, because  $\mathcal{S}(\triangleleft_{X, A}^e; M)$  is Scott-closed.  $\square$

**Lemma 111.** Let  $C$  be a Scott-closed subset of a dcpo  $X$ . Let  $W \stackrel{\text{def}}{=} \{\delta_x \mid x \in C\} \subseteq \mathcal{M}X$  and let  $\overline{W}$  be the Scott-closure of  $W$  in  $\mathcal{M}X$ . Then,  $\delta_y \in \overline{W}$  iff  $y \in C$ .

*Proof.* The “if” direction is straightforward. The “only if” direction is trivial when  $C = X$ . We now prove the case that  $C$  is a proper subset of  $X$ , and let  $U$  be the complement of  $C$ . Hence  $U$  is a nonempty Scott open subset of  $X$ . Let us assume that  $\delta_y \in \overline{W}$  but  $y \in U$ , then we know that  $[U > 0] \stackrel{\text{def}}{=} \{\nu \in \mathcal{M}X \mid \nu(U) > 0\}$  is a Scott open subset of  $\mathcal{M}X$  containing  $\delta_y$ , hence we would have that  $[U > 0] \cap W \neq \emptyset$  since by assumption  $\delta_y \in \overline{W}$ . However, this is impossible since for any  $x \in C$ ,  $\delta_x(U) = 0$ .  $\square$

**Lemma 112.** Let  $X$  be a dcpo, let  $v \in \mathbf{TD}(1, X)$  and let  $V$  be a value. Then  $v \triangleleft_{X, A}^e V$  iff  $v \overline{\triangleleft_{X, A}^e} V$ .

*Proof.* The left-to-right direction follows immediately by Lemma 106. For the other direction, we first observe that since  $v \in \mathbf{TD}(1, X)$ , then  $v \neq 0$ . Therefore by Lemma 106, it follows  $v \in \{w \mid w \triangleleft_{X, A}^e V\}$  and then by Lemma 111 we complete the proof.  $\square$

**Lemma 113.** For any value  $\cdot \vdash V : A$  and  $\triangleleft_{X, A}^e \in \text{ValRel}(X, A, e)$ , if  $m \overline{\triangleleft_{X, A}^e} V$  then  $e \circ m \leq \llbracket V \rrbracket$ .

*Proof.* We know  $m \in \mathcal{S}(\triangleleft_{X, A}^e; V) = \overline{\{v \mid v \triangleleft_{X, A}^e V\}} \cup \{0\}$  and clearly  $e \circ m \leq \llbracket V \rrbracket$  is equivalent to  $(e \circ m) \in \downarrow \llbracket V \rrbracket$ , which is a Scott-closed subset. If  $m = 0$ , then the statement is obviously true. So, assume that  $m \in \overline{\{v \mid v \triangleleft_{X, A}^e V\}}$ . Composition with  $e$  is a Scott-continuous function and therefore using Lemma 107, to finish the proof it suffices to show  $e \circ v \leq \llbracket V \rrbracket$  for each choice of  $v \triangleleft_{X, A}^e V$ . But this is true by assumption on  $\triangleleft_{X, A}^e$ .  $\square$

### C. Categories of Logical Relations

**Definition 114.** For any type  $A$ , we define a category  $\mathbf{R}(A)$  where:

- Each object is a triple  $(X, e_X, \triangleleft_X)$ , where  $X$  is a dcpo,  $e_X : X \rightarrow \llbracket A \rrbracket$  is a morphism in  $\mathbf{PD}_e$  and  $\triangleleft_X \in \text{ValRel}(X, A, e_X)$ .
- A morphism  $f : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$  is a morphism  $f : X \rightarrow Y$  in  $\mathbf{PD}_e$ , which satisfies the three additional conditions:
  - If  $v \triangleleft_X V$ , then  $f \circ v \overline{\triangleleft_Y} V$ .
  - If  $v \triangleleft_Y V$ , then  $f^p \circ v \overline{\triangleleft_X} V$ .
  - $e_X = e_Y \circ f$ .
- Composition and identities coincide with those in  $\mathbf{PD}_e$ .

**Lemma 115.** For every type  $A$ , the category  $\mathbf{R}(A)$  is indeed well-defined.

*Proof.* We have to show that  $\mathbf{id} : (X, e_X, \triangleleft_X) \rightarrow (X, e_X, \triangleleft_X)$  is indeed a morphism in  $\mathbf{R}(A)$ . This follows from Lemma 112. Next, we have to show that if  $f : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$  and  $g : (Y, e_Y, \triangleleft_Y) \rightarrow (Z, e_Z, \triangleleft_Z)$ , then we also have  $g \circ f : (X, e_X, \triangleleft_X) \rightarrow (Z, e_Z, \triangleleft_Z)$ . But this follows by Lemma 108.  $\square$

**Lemma 116.** Let  $\cdot \vdash M : A$  be a term and let  $g : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$  be a morphism in  $\mathbf{R}(A)$ . If  $m \overline{\triangleleft_X} M$  then  $g \circ m \overline{\triangleleft_Y} M$ . Moreover, if  $n \triangleleft_Y N$ , then  $g^p \circ n \overline{\triangleleft_X} N$ .

*Proof.* This follows immediately by Lemma 108.  $\square$

**Definition 117.** For every type  $A$ , we define the obvious forgetful functor  $U^A : \mathbf{R}(A) \rightarrow \mathbf{PD}_e$  by

$$\begin{aligned} U^A(X, e, \triangleleft) &= X \\ U^A(f) &= f. \end{aligned}$$

**Proposition 118.** For each type  $A$ , the category  $\mathbf{R}(A)$  has an initial object and all  $\omega$ -colimits. Furthermore, the forgetful functor  $U^A : \mathbf{R}(A) \rightarrow \mathbf{PD}_e$  preserves and reflects  $\omega$ -colimits (and also the initial objects).

*Proof.* We begin with the initial object.

**Initial object:** For any dcpo's  $X$  and  $Y$ , we write  $0_{X,Y} : X \rightarrow Y$  for the zero morphism in  $\mathbf{PD}$ . Notice that  $0_{\emptyset, X}$  is an embedding with projection counterpart given by  $0_{X, \emptyset}$ .

The object  $(\emptyset, 0_{\emptyset, \llbracket A \rrbracket}, \emptyset)$  is initial in  $\mathbf{R}(A)$ . Indeed, let  $(X, e_X, \triangleleft_X)$  be any other object of  $\mathbf{R}(A)$ . It suffices to show that  $0_{\emptyset, X} : (\emptyset, 0_{\emptyset, \llbracket A \rrbracket}, \emptyset) \rightarrow (X, e_X, \triangleleft_X)$  is a morphism in  $\mathbf{R}(A)$ , because if it exists, then it is clearly unique. The first and third conditions of Definition 114 are trivially satisfied. The second condition is also satisfied, because  $0_{\emptyset, X}^p \circ v = 0_{1, \emptyset}$ , which is the least (and only) element in  $\mathbf{DCPO}_{\mathcal{M}}(1, \emptyset)$  and this element is contained in every relation  $\overline{\triangleleft_Y}$ , including  $\overline{\emptyset}$ .

**The diagram:** For the rest of the proof, let  $D : \omega \rightarrow \mathbf{R}(A)$  be an  $\omega$ -diagram in  $\mathbf{R}(A)$ . Let  $D(i) = (X_i, e_i, \triangleleft_i)$  and let  $D(i \leq j) = f_{i,j}$ .

**Construction of the colimiting object:** Consider the  $\omega$ -diagram  $UD$  in  $\mathbf{PD}_e$ . This category has all  $\omega$ -colimits, so let  $\tau : UD \Rightarrow X_\omega$  be its colimiting cocone. Next, consider the cocone  $\epsilon : UD \Rightarrow \llbracket A \rrbracket$  defined by  $\epsilon_i \stackrel{\text{def}}{=} e_i : X_i \rightarrow \llbracket A \rrbracket$ . Let  $e_\omega : X_\omega \rightarrow \llbracket A \rrbracket$  be the unique cocone morphism  $e_\omega : \tau \rightarrow \epsilon$  induced by the colimit  $\tau$  in  $\mathbf{PD}_e$ . We now define a relation

$$\begin{aligned} \triangleleft_\omega \in \text{ValRel}(X_\omega, A, e_\omega) \text{ by:} \\ v \triangleleft_\omega V \text{ iff } \forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V. \end{aligned}$$

We have to show that  $\triangleleft_\omega \in \text{ValRel}(X_\omega, A, e_\omega)$ , as claimed above. We begin with downwards-closure. Assume  $v \triangleleft_\omega V$  and that  $v' \leq v$  in  $\mathbf{TD}(1, X_\omega)$ . Then,  $\forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V$  and therefore  $\tau_k^p \circ v' \overline{\triangleleft_k} V$ , because  $(- \overline{\triangleleft_k} V)$  is downwards-closed and so by definition  $v' \triangleleft_\omega V$ , as required.

Next, we show that  $(- \triangleleft_\omega V)$  preserves directed suprema and is therefore Scott-closed in  $\mathbf{TD}(1, X_\omega)$ . Assume that  $\{v_d\}_{d \in D}$  is a directed set, such that  $v_d \triangleleft_\omega V$  for each  $d \in D$ . Therefore,  $\forall k \in \mathbb{N}. \forall d \in D. \tau_k^p \circ v_d \overline{\triangleleft_k} V$ . Scott-closure of  $(- \overline{\triangleleft_k} V)$  implies that  $\tau_k^p \circ (\sup_{d \in D} v_d) = \sup_{d \in D} \tau_k^p \circ v_d \overline{\triangleleft_k} V$  holds for all  $k \in \mathbb{N}$ . Therefore, by definition  $\sup_{d \in D} v_d \triangleleft_\omega V$ .

We also have to show that if  $v \triangleleft_\omega V$ , then  $e_\omega \circ v \leq \llbracket V \rrbracket$ . If  $v \triangleleft_\omega V$ , then  $\forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V$  and so by Lemma 113 we get  $e_k \circ \tau_k^p \circ v \leq \llbracket V \rrbracket$ . But  $e_k \circ \tau_k^p \circ v = e_\omega \circ \tau_k \circ \tau_k^p \circ v$ . The limit-colimit coincidence theorem in the category  $\mathbf{PD}$ , shows that this forms an increasing sequence and that

$$\llbracket V \rrbracket \geq \sup_{k \in \mathbb{N}} e_\omega \circ \tau_k \circ \tau_k^p \circ v = e_\omega \circ \left( \sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \right) \circ v = e_\omega \circ \mathbf{id} \circ v = e_\omega \circ v,$$

as required. We will show that the object  $(X_\omega, e_\omega, \triangleleft_\omega)$  is the colimiting object of  $D$  in  $\mathbf{R}(A)$ . Before we can do this, we first have to construct the colimiting cocone in  $\mathbf{R}(A)$ .



**Construction of the colimiting cocone:** We show that  $\tau : D \Rightarrow X_\omega$  is a cocone in  $\mathbf{R}(A)$ . The commutativity requirements are clearly satisfied, so it suffices to show that each  $\tau_i : X_i \rightarrow X_\omega$  is a morphism  $\tau_i : (X_i, e_i, \triangleleft_i) \rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$  in  $\mathbf{R}(A)$ . Towards that end, assume that  $v \triangleleft_i V$ . We have to show that  $\tau_i \circ v \triangleleft_\omega V$ , but by Lemma 112, it suffices to show that  $\tau_i \circ v \triangleleft_k V$ <sup>8</sup>. Showing this is equivalent to showing that  $\forall k \in \mathbb{N}$ .  $\tau_k^p \circ \tau_i \circ v \triangleleft_k V$ . For any  $k \geq i$ , we get:

$$\tau_k^p \circ \tau_i \circ v = \tau_k^p \circ \tau_k \circ f_{i,k} \circ v = f_{i,k} \circ v \triangleleft_k V$$

because  $f_{i,k}$  is a morphism  $f_{i,k} : (X_i, e_i, \triangleleft_i) \rightarrow (X_k, e_k, \triangleleft_k)$  and  $v \triangleleft_i V$  by assumption. For any  $k < i$ , we get:

$$\tau_k^p \circ \tau_i \circ v = f_{k,i}^p \circ \tau_i^p \circ \tau_i \circ v = f_{k,i}^p \circ v \triangleleft_k V$$

because  $f_{k,i}$  is a morphism  $f_{k,i} : (X_k, e_k, \triangleleft_k) \rightarrow (X_i, e_i, \triangleleft_i)$  and  $v \triangleleft_i V$  by assumption (and Lemma 116).

To show that  $\tau_i : (X_i, e_i, \triangleleft_i) \rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$  is a morphism, we have to show that if  $v \triangleleft_\omega V$ , then also  $\tau_i^p \circ v \triangleleft_i V$ . But this is true by definition of  $\triangleleft_\omega$ .

Finally we have to show that  $e_i = e_\omega \circ \tau_i$ . But this is true by construction of  $e_\omega$ .

Therefore,  $\tau : D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$  is indeed a cocone of  $D$  in  $\mathbf{R}(A)$ .

**Couniversality of the cocone:** For the rest of the proof, assume that  $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$  is some other cocone of  $D$  in  $\mathbf{R}(A)$ . Next, consider the cocone  $U\alpha$  in  $\mathbf{PD}_e$  and let  $a : X_\omega \rightarrow Y$  be the unique cocone morphism  $a : U\tau \rightarrow U\alpha$  induced by the colimit in  $\mathbf{PD}_e$ . By the limit-colimit coincidence theorem in  $\mathbf{PD}$ , we get

$$a = a \circ \mathbf{id} = a \circ \sup_{i \in \mathbb{N}} \tau_i \circ \tau_i^p = \sup_{i \in \mathbb{N}} a \circ \tau_i \circ \tau_i^p = \sup_{i \in \mathbb{N}} \alpha_i \circ \tau_i^p$$

We will show that  $a : (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$  is a morphism in  $\mathbf{R}(A)$ . Towards this end, assume that  $v \triangleleft_\omega V$ . Then  $\forall k \in \mathbb{N}$ .  $\tau_k^p \circ v \triangleleft_k V$  and therefore  $\alpha_k \circ \tau_k^p \circ v \triangleleft_Y V$ , because by assumption  $\alpha_k : (X_k, e_k, \triangleleft_k) \rightarrow (Y, e_Y, \triangleleft_Y)$ . Since  $(-\triangleleft_Y V)$  is closed under suprema, it follows

$$\sup_{k \in \mathbb{N}} \alpha_k \circ \tau_k^p \circ v = \left( \sup_{k \in \mathbb{N}} \alpha_k \circ \tau_k^p \right) \circ v = a \circ v \triangleleft_Y V,$$

which shows that  $a$  satisfies one of the requirements for being a morphism in  $\mathbf{R}(A)$ .

For the second requirement, assume that  $v \triangleleft_Y V$ . Then,  $\forall k \in \mathbb{N}$ .  $\alpha_k^p \circ v \triangleleft_k V$ , by assumption on  $\alpha_k$ . The same argument shows that  $\forall k \in \mathbb{N}$ .  $\tau_k \circ \alpha_k^p \circ v \triangleleft_\omega V$ , because  $\tau_k$  is also a morphism in the category. Since  $(-\triangleleft_\omega V)$  is closed under suprema, we get:

$$\sup_{k \in \mathbb{N}} \tau_k \circ \alpha_k^p \circ v = \sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \circ a^p \circ v = \left( \sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \right) \circ a^p \circ v = a^p \circ v \triangleleft_\omega V$$

as required.

For the third requirement, we have to show that  $e_\omega = e_Y \circ a$ . By assumption on the cone  $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$ , we have that  $\forall i \in \mathbb{N}$ .  $e_i = e_Y \circ \alpha_i$  and by construction of  $a$ , we know  $\alpha_i = a \circ \tau_i$ . Therefore  $\forall i \in \mathbb{N}$ .  $e_i = e_Y \circ a \circ \tau_i$ . However,  $e_\omega$  is by construction the unique morphism in  $\mathbf{PD}_e$ , such that  $\forall i$ .  $e_i = e_\omega \circ \tau_i$ , which shows that  $e_\omega = e_Y \circ a$ , as required. Therefore, we have shown that  $a : (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$  is indeed a morphism in  $\mathbf{R}(A)$ .

That  $a : \tau \rightarrow \alpha$  is the unique cocone morphism is now obvious, because if  $a' : \tau \rightarrow \alpha$  is another one, then  $Ua$  and  $Ua'$  are both cocone morphisms between  $U\tau$  and  $U\alpha$  in  $\mathbf{PD}_e$  and therefore  $a = Ua = Ua' = a'$ . Therefore,  $\tau : D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$  is indeed the colimiting cocone of  $D$  in  $\mathbf{R}(A)$ , which shows that  $\mathbf{R}(A)$  has all  $\omega$ -colimits.

**$U^A$  preserves  $\omega$ -colimits:** Assume that the cocone  $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$  from above is colimiting in  $\mathbf{R}(A)$ . But, we know that  $\tau : D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$  is also a colimiting cocone of  $D$ . Therefore, there exists a unique cocone isomorphism  $i : \tau \rightarrow \alpha$ . Then,  $Ui : U\tau \rightarrow U\alpha$  is a cocone isomorphism in  $\mathbf{PD}_e$ . However, by construction,  $U\tau$  is a colimiting cocone of  $UD$  in  $\mathbf{PD}_e$  and therefore so is  $U\alpha$ .

**$U^A$  reflects  $\omega$ -colimits:** Assume that the cocone  $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$  from above is such that  $U\alpha : UD \Rightarrow Y$  is colimiting in  $\mathbf{PD}_e$ . Then the morphism  $a : X_\omega \rightarrow Y$  from above is an isomorphism in  $\mathbf{PD}_e$ . We have already shown that  $a : (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$  is a morphism in  $\mathbf{R}(A)$ . Thus, to finish the proof, it suffices to show that  $a^{-1}$  is a morphism in  $\mathbf{R}(A)$  in the opposite direction. But this is obviously true, because  $a^{-1} = a^p$  and  $(a^{-1})^p = a$  and we have shown above that these morphisms satisfy the logical requirements and clearly  $e_Y = e_\omega \circ a^{-1}$ . □

Next, we introduce important relation constructors and some new notation.

**Notation 119.** Given morphisms  $m_i : 1 \rightarrow X_i$ , for  $i \in \{1, \dots, n\}$ , we define

$$\langle\langle m_1, \dots, m_n \rangle\rangle \stackrel{\text{def}}{=} (m_1 \dot{\times} \dots \dot{\times} m_n) \circ \mathcal{J}\langle \text{id}_1, \dots, \text{id}_1 \rangle : 1 \rightarrow X_1 \times \dots \times X_n.$$

<sup>8</sup>Note that  $\tau_i \circ v$  is a morphism of  $\mathbf{TD}$ , because  $v$  is one and because  $\tau_i \in \mathbf{PD}_e$  which is a subcategory of  $\mathbf{TD}$ .

**Notation 120.** Given morphisms  $x : 1 \rightarrow X$  and  $f : 1 \rightarrow [X \rightarrow Y]$  in  $\mathbf{DCPO}_{\mathcal{M}}$ , let  $f[x] : 1 \rightarrow Y$  be the morphism defined by

$$f[x] \stackrel{\text{def}}{=} \epsilon \circ (f \dot{\times} x) \circ \mathcal{J}\langle \text{id}_1, \text{id}_1 \rangle.$$

**Definition 121 (Relation Constructions).** We define relation constructors:

- If  $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$  and  $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$ , define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} + \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}(X_1 + X_2, A_1 + A_2, e_1 \dot{+} e_2) \text{ by:} \\ & \mathcal{J} \text{in}_i \circ v \langle \triangleleft_{X_1, A_1}^{e_1} + \triangleleft_{X_2, A_2}^{e_2} \rangle \text{ in}_i V \text{ iff } v \triangleleft_{X_i, A_i}^{e_i} V \quad (\text{for } i \in \{1, 2\}). \end{aligned}$$

- If  $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$  and  $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$ , define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} \times \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}(X_1 \times X_2, A_1 \times A_2, e_1 \dot{\times} e_2) \text{ by:} \\ & \langle\langle v_1, v_2 \rangle\rangle \langle \triangleleft_{X_1, A_1}^{e_1} \times \triangleleft_{X_2, A_2}^{e_2} \rangle (V_1, V_2) \text{ iff } v_1 \triangleleft_{X_1, A_1}^{e_1} V_1 \text{ and } v_2 \triangleleft_{X_2, A_2}^{e_2} V_2. \end{aligned}$$

- If  $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$  and  $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$ , define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} \rightarrow \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}([X_1 \rightarrow X_2], A_1 \rightarrow A_2, \mathcal{J}[e_1^p \rightarrow e_2]) \text{ by:} \\ & f \langle \triangleleft_{X_1, A_1}^{e_1} \rightarrow \triangleleft_{X_2, A_2}^{e_2} \rangle \lambda x. M \text{ iff } \mathcal{J}[e_1^p \rightarrow e_2] \circ f \leq \llbracket \lambda x. M \rrbracket \text{ and } \forall (v \triangleleft_{X_1, A_1}^{e_1} V). f[v] \overline{\triangleleft_{X_2, A_2}^{e_2}} (\lambda x. M)V. \end{aligned}$$

**Lemma 122.** *The assignments in Definition 121 are indeed well-defined.*

*Proof.* Straightforward verification. □

Next, a simple lemma that we use later.

**Lemma 123.** *Assume we are given morphisms  $f : 1 \rightarrow [C \rightarrow D]$ ,  $h : A \rightarrow C$ ,  $g : D \rightarrow B$  and  $v : 1 \rightarrow A$ . Then*

$$(\mathcal{J}[h \rightarrow g] \circ f)[v] = g \circ f[h \circ v].$$

*Proof.*

$$\begin{aligned} (\mathcal{J}[h \rightarrow g] \circ f)[v] &= \epsilon \circ ((\mathcal{J}[h \rightarrow g] \circ f) \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle && \text{(Definition)} \\ &= \epsilon \circ (\mathcal{J}[h \rightarrow g] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \epsilon \circ (\mathcal{J}[\mathbf{id} \rightarrow g] \dot{\times} \mathbf{id}) \circ (\mathcal{J}[h \rightarrow \mathbf{id}] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= g \circ \epsilon \circ (\mathcal{J}[h \rightarrow \mathbf{id}] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle && \text{(Naturality of } \epsilon) \\ &= g \circ \epsilon \circ (\mathbf{id} \dot{\times} h) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle && \text{(Parameterised adjunction [31, pp.102])} \\ &= g \circ f[h \circ v] && \text{(Definition)} \end{aligned}$$

□

**Notation 124.** Throughout the rest of the paper we shall write  $(- \rightarrow_e -) \stackrel{\text{def}}{=} [- \rightarrow -]_e^{\mathcal{J}} : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$ . That is, we just introduce a more concise notation for the functor  $[- \rightarrow -]_e^{\mathcal{J}}$  from Proposition 71.

The next definition is crucial. Given two logical relations, it is used to define the product, coproduct and function space logical relations. Moreover, this is done in a functorial sense on the categories  $\mathbf{R}(A)$ .

**Definition 125.** Let  $A$  and  $B$  be types. We define *covariant* functors in the following way (recall Definition 121):

- 1)  $\times^{A, B} : \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \times B)$  by

$$\begin{aligned} (X, e_X, \triangleleft_X) \times^{A, B} (Y, e_Y, \triangleleft_Y) &\stackrel{\text{def}}{=} (X \times Y, e_X \dot{\times}_e e_Y, \triangleleft_X \times \triangleleft_Y) \\ f \times^{A, B} g &\stackrel{\text{def}}{=} f \dot{\times}_e g \end{aligned}$$

- 2)  $+^{A, B} : \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A + B)$  by

$$\begin{aligned} (X, e_X, \triangleleft_X) +^{A, B} (Y, e_Y, \triangleleft_Y) &\stackrel{\text{def}}{=} (X + Y, e_X \dot{+}_e e_Y, \triangleleft_X + \triangleleft_Y) \\ f +^{A, B} g &\stackrel{\text{def}}{=} f \dot{+}_e g \end{aligned}$$

- 3)  $\rightarrow^{A, B} : \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \rightarrow B)$  by

$$\begin{aligned} (X, e_X, \triangleleft_X) \rightarrow^{A, B} (Y, e_Y, \triangleleft_Y) &\stackrel{\text{def}}{=} ([X \rightarrow Y], e_X \rightarrow_e e_Y, \triangleleft_X \rightarrow \triangleleft_Y) \\ f \rightarrow^{A, B} g &\stackrel{\text{def}}{=} f \rightarrow_e g \end{aligned}$$

**Proposition 126.** *Each of the functors from Definition 125 is well-defined.*

*Proof.* We will show the case for function types which is the most complicated. The other cases follow by a straightforward verification using similar arguments.

**Function types:** Let

$$\begin{aligned} f_1 &: (X_1, e_1^X, \triangleleft_1^X) \rightarrow (Y_1, e_1^Y, \triangleleft_1^Y) \\ f_2 &: (X_2, e_2^X, \triangleleft_2^X) \rightarrow (Y_2, e_2^Y, \triangleleft_2^Y) \end{aligned}$$

We have to show

$$f_1 \rightarrow_e f_2 : (X_1 \rightarrow_e X_2, e_1^X \rightarrow_e e_2^X, \triangleleft_1^X \rightarrow \triangleleft_2^X) \rightarrow (Y_1 \rightarrow_e Y_2, e_1^Y \rightarrow_e e_2^Y, \triangleleft_1^Y \rightarrow \triangleleft_2^Y)$$

is a morphism in  $\mathbf{R}(A \rightarrow B)$ .

First, we show that  $f_1 \rightarrow_e f_2$  respects the embedding component. Indeed:

$$e_1^X \rightarrow_e e_2^X = (e_1^Y \circ f_1) \rightarrow_e (e_2^Y \circ f_2) = (e_1^Y \rightarrow_e e_2^Y) \circ (f_1 \rightarrow_e f_2).$$

Next, assume that  $v \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$ . Assume further that  $v' \triangleleft_1^Y V'$ . Then, clearly  $f_1^p \circ v' \triangleleft_1^X V'$ . If  $f_1^p \circ v' = 0$ , then it trivially follows that  $v[f_1^p \circ v'] = 0 \triangleleft_2^X VV'$ . Otherwise,  $f_1^p \circ v' \in \mathbf{TD}$  and so  $f_1^p \circ v' \triangleleft_1^X V'$  and therefore  $v[f_1^p \circ v'] \triangleleft_2^X VV'$ . In all cases,  $v[f_1^p \circ v'] \triangleleft_2^X VV'$  and therefore  $f_2 \circ v[f_1^p \circ v'] \triangleleft_2^Y VV'$ . But then, by Lemma 123 we have:

$$f_2 \circ v[f_1^p \circ v'] = (\mathcal{J}[f_1^p \rightarrow f_2] \circ v)[v'] = ((f_1 \rightarrow_e f_2) \circ v)[v'] \triangleleft_2^Y VV'.$$

Furthermore

$$(e_1^Y \rightarrow_e e_2^Y) \circ (f_1 \rightarrow_e f_2) \circ v = (e_1^X \rightarrow_e e_2^X) \circ v \leq \llbracket V \rrbracket$$

and therefore by definition  $(f_1 \rightarrow_e f_2) \circ v \langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle V$  and therefore also  $(f_1 \rightarrow_e f_2) \circ v \langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle V$ , as required.

For the other direction, assume that  $v \langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle V$ . Assume further that  $v' \triangleleft_1^X V'$ . Then, clearly  $f_1 \circ v' \triangleleft_1^Y V'$ . If  $f_1 \circ v' = 0$ , then it trivially follows that  $v[f_1 \circ v'] = 0 \triangleleft_2^Y VV'$ . Otherwise,  $f_1 \circ v' \in \mathbf{TD}$  and so  $f_1 \circ v' \triangleleft_1^Y V'$  and therefore  $v[f_1 \circ v'] \triangleleft_2^Y VV'$ . In all cases,  $v[f_1 \circ v'] \triangleleft_2^Y VV'$  and therefore  $f_2^p \circ v[f_1 \circ v'] \triangleleft_2^X VV'$ . But then, by Lemma 123 we have:

$$f_2^p \circ v[f_1 \circ v'] = (\mathcal{J}[f_1 \rightarrow f_2^p] \circ v)[v'] = ((f_1 \rightarrow_e f_2)^p \circ v)[v'] \triangleleft_2^X VV'.$$

Furthermore

$$\begin{aligned} (e_1^X \rightarrow_e e_2^X) \circ (f_1 \rightarrow_e f_2)^p \circ v &= \mathcal{J}[(e_1^X)^p \rightarrow e_2^X] \circ \mathcal{J}[f_1 \rightarrow f_2^p] \circ v \\ &= \mathcal{J}[(f_1 \circ (e_1^X)^p) \rightarrow (e_2^X \circ f_2^p)] \circ v \\ &\leq \mathcal{J}[(f_1 \circ (e_1^X)^p) \rightarrow e_2^Y] \circ v \\ &\leq \mathcal{J}[(e_1^Y)^p \rightarrow e_2^Y] \circ v \\ &\leq \llbracket V \rrbracket. \end{aligned}$$

If  $(f_1 \rightarrow_e f_2)^p \circ v \in \mathbf{TD}$ , then  $(f_1 \rightarrow_e f_2)^p \circ v \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$  by definition. Otherwise,  $(f_1 \rightarrow_e f_2)^p \circ v = 0$  and then trivially  $(f_1 \rightarrow_e f_2)^p \circ v = 0 \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$ . Therefore, in all cases  $(f_1 \rightarrow_e f_2)^p \circ v \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$ , as required.

Therefore, the functor  $\rightarrow^{A,B}$  is indeed well-defined.  $\square$

Observe that Definition 125 lifts the functors that we use to interpret our types in the category  $\mathbf{DCPO}_{\mathcal{M}}$  to the categories  $\mathbf{R}(A)$ . Next, we show that the functors we just defined are also suitable for forming (parameterised) initial algebras.

**Proposition 127.** *For  $\star \in \{\times, +, \rightarrow\}$ , for all types  $A$  and  $B$ , the functor  $\star^{A,B} : \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \star B)$  is  $\omega$ -cocontinuous and the following diagram:*

$$\begin{array}{ccc} \mathbf{R}(A) \times \mathbf{R}(B) & \xrightarrow{\star^{A,B}} & \mathbf{R}(A \star B) \\ \downarrow U^A \times U^B & & \downarrow U^{A \star B} \\ \mathbf{PD}_e \times \mathbf{PD}_e & \xrightarrow{\star_e} & \mathbf{PD}_e \end{array}$$

commutes.

*Proof.* Commutativity of the diagram is immediate from the definitions. To see  $\omega$ -cocontinuity, let  $D$  be an  $\omega$ -diagram in  $\mathbf{R}(A) \times \mathbf{R}(B)$  and let  $\tau$  be its colimiting cocone. Because the functors  $U^A, U^B$  and  $\star_e$  are  $\omega$ -cocontinuous, it follows that :

$$\begin{aligned} & (\star_e \circ U^A \times U^B)\tau \text{ is colimiting in } \mathbf{PD}_e \\ \implies & (U^{A \star B} \circ \star^{A,B})\tau \text{ is colimiting in } \mathbf{PD}_e && \text{(Commutativity of the above diagram)} \\ \implies & \star^{A,B}\tau \text{ is colimiting in } \mathbf{R}(A \star B) && (U \text{ reflects } \omega\text{-colimits}) \end{aligned}$$

which shows that  $\star^{A,B}$  is  $\omega$ -cocontinuous.  $\square$

Next, we establish an isomorphism between the categories  $\mathbf{R}(\mu X.A)$  and  $\mathbf{R}(A[\mu X.A/X])$ .

**Definition 128.** We define constructors for folding and unfolding logical relations as follows:

- If  $\triangleleft_{X,A[\mu Y.A/Y]}^e \in \text{ValRel}(X, A[\mu Y.A/Y], e)$ , define

$$\begin{aligned} & (\mathbb{I}^{\mu Y.A} \triangleleft_{X,A[\mu Y.A/Y]}^e) \in \text{ValRel}(X, \mu Y.A, \text{fold} \circ e) \text{ by:} \\ v (\mathbb{I}^{\mu Y.A} \triangleleft_{X,A[\mu Y.A/Y]}^e) \text{ fold } V \text{ iff } v \triangleleft_{X,A[\mu Y.A/Y]}^e V. \end{aligned}$$

- If  $\triangleleft_{X,\mu Y.A}^e \in \text{ValRel}(X, \mu Y.A, e)$ , define

$$\begin{aligned} & (\mathbb{E}^{\mu Y.A} \triangleleft_{X,\mu Y.A}^e) \in \text{ValRel}(X, A[\mu Y.A/Y], \text{unfold} \circ e) \text{ by:} \\ v (\mathbb{E}^{\mu Y.A} \triangleleft_{X,\mu Y.A}^e) V \text{ iff } v \triangleleft_{X,\mu Y.A}^e \text{ fold } V. \end{aligned}$$

**Proposition 129.** *The above assignments are indeed well-defined.*

*Proof.* Straightforward verification.  $\square$

**Proposition 130.** *For every type  $\cdot \vdash \mu X.A$ , we have an isomorphism of categories*

$$\mathbb{I}^{\mu X.A} : \mathbf{R}(A[\mu X.A/X]) \cong \mathbf{R}(\mu X.A) : \mathbb{E}^{\mu X.A},$$

where the functors are defined by

$$\begin{aligned} \mathbb{I}^{\mu X.A} : \mathbf{R}(A[\mu X.A/X]) &\rightarrow \mathbf{R}(\mu X.A) & \mathbb{E}^{\mu X.A} : \mathbf{R}(\mu X.A) &\rightarrow \mathbf{R}(A[\mu X.A/X]) \\ \mathbb{I}^{\mu X.A}(Y, e, \triangleleft) &= (Y, \text{fold} \circ e, \mathbb{I}^{\mu X.A} \triangleleft) & \mathbb{E}^{\mu X.A}(Y, e, \triangleleft) &= (Y, \text{unfold} \circ e, \mathbb{E}^{\mu X.A} \triangleleft) \\ \mathbb{I}^{\mu X.A}(f) &= f & \mathbb{E}^{\mu X.A}(f) &= f \end{aligned}$$

*Proof.* The proof is essentially the same as [16, Lemma 7.23], with one extra proof obligation, namely we have to show that our functorial assignments respect the embedding components. But this is obviously true.  $\square$

This finishes the categorical development of the categories  $\mathbf{R}(A)$ .

#### D. Augmented Interpretation of Types

We have now established sufficient categorical structure in order to construct parameterised initial algebras in the categories  $\mathbf{R}(A)$ . Furthermore, we have sufficient structure to also define an *augmented* interpretation of types in these categories. The main idea behind providing the augmented interpretation is to show how to pick out the logical relations we need from all those that exist in the categories  $\mathbf{R}(A)$ .

**Notation 131.** Given any type context  $\Theta = X_1, \dots, X_n$  and closed types  $\cdot \vdash C_i$  with  $i \in \{1, \dots, n\}$ , we shall write  $\vec{C}$  for  $C_1, \dots, C_n$  and we also write  $[\vec{C}/\Theta]$  for  $[C_1/X_1, \dots, C_n/X_n]$ .

**Definition 132.** For any type  $\Theta \vdash A$  and closed types  $\vec{C}$ , we define their *augmented interpretation* to be the functor

$$\|\Theta \vdash A\|^{\vec{C}} : \mathbf{R}(C_1) \times \dots \times \mathbf{R}(C_n) \rightarrow \mathbf{R}(A[\vec{C}/\Theta])$$

defined by induction on the derivation of  $\Theta \vdash A$ :

$$\begin{aligned} \|\Theta \vdash \Theta_i\|^{\vec{C}} &:= \Pi_i \\ \|\Theta \vdash A \star B\|^{\vec{C}} &:= \star^{A[\vec{C}/\Theta], B[\vec{C}/\Theta]} \circ (\|\Theta \vdash A\|^{\vec{C}}, \|\Theta \vdash B\|^{\vec{C}}) && \text{(for } \star \in \{+, \times, \rightarrow\}) \\ \|\Theta \vdash \mu X.A\|^{\vec{C}} &:= \left( \mathbb{I}^{\mu X.A[\vec{C}/\Theta]} \circ \|\Theta, X \vdash A\|^{\vec{C}, \mu X.A[\vec{C}/\Theta]} \right)^\sharp, \end{aligned}$$

where the  $(-)^{\sharp}$  operation is from Definition 41.

**Proposition 133.** *Each functor  $\|\Theta \vdash A\|^{\vec{C}}$  is well-defined and  $\omega$ -cocontinuous. Moreover, the following diagram:*

$$\begin{array}{ccc}
\mathbf{R}(C_1) \times \cdots \times \mathbf{R}(C_n) & \xrightarrow{\|\Theta \vdash A\|^{\vec{C}}} & \mathbf{R}(A[\vec{C}/\Theta]) \\
\downarrow U^{C_1} \times \cdots \times U^{C_n} & & \downarrow U^{A[\vec{C}/\Theta]} \\
\mathbf{PD}_e \times \cdots \times \mathbf{PD}_e & \xrightarrow{\llbracket \Theta \vdash A \rrbracket} & \mathbf{PD}_e
\end{array}$$

*commutes.*

*Proof.* The proof is essentially the same as [16, Proposition 7.26]. □

Next, a corollary which shows that parameterised initial algebras for our type expressions are constructed in the same way in both categories.

**Corollary 134.** *The 2-categorical diagram:*

$$\begin{array}{ccc}
& \mathbb{I}^{\mu X.A[\vec{C}/\Theta]} \circ \|\Theta, X \vdash A\|^{\vec{C}, \mu X.A[\vec{C}/\Theta]} \circ \langle \text{Id}, \|\Theta \vdash \mu X.A\|^{\vec{C}} \rangle & \\
& \swarrow & \searrow \\
\mathbf{R}(C_1) \times \cdots \times \mathbf{R}(C_n) & \begin{array}{c} \Downarrow \iota \\ \|\Theta \vdash \mu X.A\|^{\vec{C}} \end{array} & \mathbf{R}(A[\vec{C}/\Theta]) \\
\downarrow U^{C_1} \times \cdots \times U^{C_n} & & \downarrow U^{A[\vec{C}/\Theta]} \\
\mathbf{PD}_e \times \cdots \times \mathbf{PD}_e & \begin{array}{c} \Downarrow \iota \\ \llbracket \Theta \vdash \mu X.A \rrbracket \end{array} & \mathbf{PD}_e \\
& \swarrow \llbracket \Theta, X \vdash A \rrbracket \circ \langle \text{Id}, \llbracket \Theta \vdash \mu X.A \rrbracket \rangle & \searrow \\
& \mathbb{I}^{\mu X.A} \circ \llbracket \Theta \vdash \mu X.A \rrbracket & 
\end{array}$$

*commutes, where  $\iota$  is the parameterised initial algebra isomorphism (see Definition 41).*

*Proof.* The proof is the same as [16, Corollary 7.27]. □

Proposition 133 shows that the first component of the augmented interpretation coincides with the standard interpretation. This is true for all types, including open ones. In the special case for closed types, let  $\|A\| \stackrel{\text{def}}{=} \|\cdot \vdash A\|(\ast)$ , where  $\ast$  is the unique object of the terminal category  $\mathbf{1} = \mathbf{R}(A)^0$ . Proposition 133 therefore shows that  $U\|A\| = \llbracket A \rrbracket$ , which means that  $\|A\|$  has the form  $\|A\| = (\llbracket A \rrbracket, e, \triangleleft)$ , where  $e : \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket$  is some embedding. Next, we show that  $e = \mathbf{id}$ . In order to do this, we prove a stronger proposition first. We show that the action of the functor  $\|\Theta \vdash A\|^{\vec{C}}$  on the embedding component is also completely determined by the action of  $\llbracket \Theta \vdash A \rrbracket$  on embeddings.

**Proposition 135.** *For every functor  $\|\Theta \vdash A\|^{\vec{C}}$  and objects  $(X_i, e_i, \triangleleft_i)$  with  $i \in \{1, \dots, n\}$ , we have:*

$$\pi_e \left( \|\Theta \vdash A\|^{\vec{C}} \left( (X_1, e_1, \triangleleft_1), \dots, (X_n, e_n, \triangleleft_n) \right) \right) = \llbracket \Theta \vdash A \rrbracket (e_1, \dots, e_n),$$

*where for an object  $(Z, e_Z, \triangleleft_Z)$  in any category  $\mathbf{R}(B)$ , we define  $\pi_e(Z, e_Z, \triangleleft_Z) = e_Z$ .*

*Proof.* By induction on the derivation of  $\Theta \vdash A$ .

**Case  $\Theta_i$ :** This is obviously true.

**Case  $A = A_1 \star A_2$ , for  $\star \in \{\times, +, \rightarrow\}$ :** The statement follows easily by induction and the fact that for every pair of objects  $(Y, e_Y, \triangleleft_Y)$  and  $(Z, e_Z, \triangleleft_Z)$  we have

$$\pi_e \left( (Y, e_Y, \triangleleft_Y) \star^{A_1, A_2} (Z, e_Z, \triangleleft_Z) \right) = e_Y \star_e e_Z$$

which follows by definition of the relevant functors.

**Case  $\mu X.A$ :** First we introduce some abbreviations to simplify notation. We define:

- $T \stackrel{\text{def}}{=} \|\Theta, X \vdash A\|^{\vec{C}, \mu X.A[\vec{C}/\Theta]}$ .
- $H \stackrel{\text{def}}{=} \llbracket \Theta, X \vdash A \rrbracket$ .
- $\mathbb{I} \stackrel{\text{def}}{=} \llbracket \mu X.A[\vec{C}/\Theta] \rrbracket$ .
- $\overrightarrow{(X, e, \triangleleft)} \stackrel{\text{def}}{=} ((X_1, e_1, \triangleleft_1), \dots, (X_n, e_n, \triangleleft_n))$ .
- $\vec{X} \stackrel{\text{def}}{=} (X_1, \dots, X_n)$ .
- $\vec{e} \stackrel{\text{def}}{=} (e_1, \dots, e_n)$ .

Now, let  $(Y, e_Y, \triangleleft_Y) \stackrel{\text{def}}{=} (\mathbb{I} \circ T)^{\sharp}(\overrightarrow{(X, e, \triangleleft)})$ . To finish the proof, we have to show that  $H^{\sharp}(\vec{e}) = e_Y$ . From Proposition 133 we know that  $Y = H^{\sharp}(\vec{X})$ . From Corollary 134, we have a parameterised initial algebra isomorphism

$$\iota: \mathbb{I}T \left( \overrightarrow{(X, e, \triangleleft)}, (H^{\sharp}\vec{X}, e_Y, \triangleleft_Y) \right) \rightarrow (H^{\sharp}\vec{X}, e_Y, \triangleleft_Y) \quad (9)$$

which is also a parameterised initial algebra isomorphism

$$\iota: H \left( \vec{X}, H^{\sharp}\vec{X} \right) \rightarrow H^{\sharp}\vec{X} \quad (10)$$

in  $\mathbf{PD}_e$ . By the induction hypothesis for  $T$  and  $H$  and Proposition 133, we get

$$T \left( \overrightarrow{(X, e, \triangleleft)}, (H^{\sharp}\vec{X}, e_Y, \triangleleft_Y) \right) = \left( H(\vec{X}, H^{\sharp}\vec{X}), H(\vec{e}, e_Y), \blacktriangleleft \right),$$

where  $\blacktriangleleft$  is some (unimportant) logical relation. Therefore by (9) and definition of  $\mathbb{I}$ , we get that

$$\iota: \left( H(\vec{X}, H^{\sharp}\vec{X}), \text{fold} \circ H(\vec{e}, e_Y), \mathbb{I} \blacktriangleleft \right) \rightarrow (H^{\sharp}\vec{X}, e_Y, \triangleleft_Y) \quad (11)$$

is an isomorphism with the indicated type. This means that in the category  $\mathbf{PD}_e$ , we have:

$$\text{fold} \circ H(\vec{e}, e_Y) = e_Y \circ \iota \quad (12)$$

where we already know that  $\iota = \iota_{X_1, \dots, X_n}$  is the parameterised initial algebra in  $\mathbf{PD}_e$  of  $H$ . But, by definition, so is  $\text{fold}$  and in fact  $\text{fold} = \iota_{\llbracket C_1 \rrbracket, \dots, \llbracket C_n \rrbracket}$ . However,  $H^{\sharp}\vec{e}$  is the unique morphism, such that

$$\iota_{\llbracket C_1 \rrbracket, \dots, \llbracket C_n \rrbracket} \circ H(\vec{e}, H^{\sharp}\vec{e}) = H^{\sharp}\vec{e} \circ \iota_{X_1, \dots, X_n}$$

which is the universal property of a parameterised initial algebra (see [16, Remark 4.6]) and therefore by equation (12) it follows that  $e_Y = H^{\sharp}\vec{e}$ , as required.  $\square$

**Corollary 136.** *For every closed type  $A$ , we have  $\|A\| = (\llbracket A \rrbracket, \mathbf{id}_{\llbracket A \rrbracket}, \triangleleft_A)$  for some logical relation  $\triangleleft_A$ .*

*Proof.* We already know that the first component is  $\llbracket A \rrbracket$ . For the second component, the previous proposition shows that  $\pi_e \|A\| = \pi_e \|\cdot \vdash A\|(\ast) = \llbracket \cdot \vdash A \rrbracket(\mathbf{id}_{\ast}) = \mathbf{id}_{\llbracket A \rrbracket}$ , where  $\ast$  denotes the empty tuple of objects and  $\mathbf{id}_{\ast}$  the empty tuple of embeddings.  $\square$

Finally, we want to show that the third component of  $\|A\|$  is the logical relation that we need to carry out the adequacy proof. For this, we have to prove a substitution lemma first.

**Lemma 137** (Substitution). *For any types  $\Theta, X \vdash A$  and  $\Theta \vdash B$  and closed types  $C_1, \dots, C_n$ , we have:*

$$\|\Theta \vdash A[B/X]\|^{\vec{C}} = \|\Theta, X \vdash A\|^{\vec{C}, B[\vec{C}/\Theta]} \circ \langle \text{Id}, \|\Theta \vdash B\|^{\vec{C}} \rangle.$$

*Proof.* The proof is the same as [16, Lemma 7.30].  $\square$

For each type  $A$ , we have now provided an augmented interpretation  $\|A\|$  of  $A$  in the category  $\mathbf{R}(A)$ . The interpretation  $\|-\|$  satisfies all the fundamental properties of  $\llbracket - \rrbracket$ , as we have now shown. It should now be clear that this augmented interpretation is true to its name, because it carries strictly more information compared to the standard interpretation of types. The additional information that  $\|A\|$  carries is precisely the logical relation that we need at type  $A$ , as we show in the next subsection.

### E. Existence of the Logical Relations

We can now show that the logical relations we need for the adequacy proof exist.

**Theorem 138.** *For each closed type  $A$ , there exist formal approximation relations:*

$$\begin{aligned} \triangleleft_A &\subseteq \mathbf{TD}(1, \llbracket A \rrbracket) \times \text{Val}(A) \\ \overline{\triangleleft}_A &\subseteq \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \times \text{Prog}(A) \end{aligned}$$

which satisfy the following properties:

(A1)  $\mathcal{J}\text{in}_i \circ v \triangleleft_{A_1+A_2} \text{in}_i V$  iff  $v \triangleleft_{A_i} V$ , where  $i \in \{1, 2\}$ .

(A2)  $\langle\langle v_1, v_2 \rangle\rangle \triangleleft_{A_1 \times A_2} (V_1, V_2)$  iff  $v_1 \triangleleft_{A_1} V_1$  and  $v_2 \triangleleft_{A_2} V_2$ .

(A3)  $f \triangleleft_{A \rightarrow B} \lambda x. M$  iff  $f \leq \llbracket \lambda x. M \rrbracket$  and  $\forall (v \triangleleft_A V). f[v] \overline{\triangleleft}_B (\lambda x. M)V$ .

(A4)  $v \triangleleft_{\mu X.A} \text{fold } V$  iff  $\text{unfold} \circ v \triangleleft_{A[\mu X.A/X]} V$ .

(B)  $m \overline{\triangleleft}_A M$  iff  $m \in \mathcal{S}(\triangleleft_A; M)$ , where  $\mathcal{S}(\triangleleft_A; M)$  is the Scott-closure in  $\mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket)$  of the set

$$\mathcal{S}_0(\triangleleft_A; M) \stackrel{\text{def}}{=} \left\{ \sum_{\pi \in F} P(\pi)v_\pi \mid F \subseteq \text{TPaths}(M), F \text{ is finite and } v_\pi \triangleleft_A V_\pi \text{ for each } \pi \in F \right\} \text{ (see Definition 103).}$$

(C1) If  $v \triangleleft_A V$ , then  $v \leq \llbracket V \rrbracket$ .

(C2)  $(- \triangleleft_A V)$  is a Scott-closed subset of  $\mathbf{TD}(1, \llbracket A \rrbracket)$ .

(C3) If  $m \overline{\triangleleft}_A M$ , then  $m \leq \llbracket M \rrbracket$ .

(C4)  $(- \overline{\triangleleft}_A M)$  is a Scott-closed subset of  $\mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket)$ .

(C5) If  $v \in \mathbf{TD}(1, \llbracket A \rrbracket)$  and  $V$  is a value, then  $v \triangleleft_A V$  iff  $v \overline{\triangleleft}_A V$ .

*Proof.* Consider the object  $\llbracket A \rrbracket \in \mathbf{R}(A)$ . We have already shown that  $\llbracket A \rrbracket = (\llbracket A \rrbracket, \mathbf{id}_{\llbracket A \rrbracket}, \triangleleft_A)$  for some logical relation  $\triangleleft_A \in \text{ValRel}(\llbracket A \rrbracket, A, \mathbf{id}_{\llbracket A \rrbracket})$ . We now show that  $\triangleleft_A$  satisfies the required properties. Notice that the embedding components are just identities.

Property (B) is satisfied by construction (Definition 109). Properties (C1) and (C2) are also satisfied by construction (Definition 101). Property (C4) is satisfied by construction and property (B). Property (C3) is satisfied, because if  $m \overline{\triangleleft}_A M$ , then by Corollary 98 and property (C1) it follows that  $\mathcal{S}_0(\triangleleft_A; M) \subseteq \downarrow \llbracket M \rrbracket$ . The latter set is Scott-closed and therefore  $m \in \mathcal{S}(\triangleleft_A; M) \subseteq \downarrow \llbracket M \rrbracket$ , as required. Property (C5) is satisfied by Lemma 112.

Properties (A1), (A2) and (A3) are satisfied, because for  $\star \in \{+, \times, \rightarrow\}$ , we have that  $\triangleleft_{A \star B} = \triangleleft_A \star \triangleleft_B$  and then by Definition 121.

To show that property (A4) is also satisfied, we reason as follows. Consider the isomorphism

$$\text{unfold}_{\mu X.A} : \llbracket \mu X.A \rrbracket \cong \llbracket X \vdash A \rrbracket \llbracket \mu X.A \rrbracket = \llbracket A[\mu X.A/X] \rrbracket : \text{fold}_{\mu X.A}$$

from Definition 59. By Corollary 134 and Lemma 137 (when  $\Theta = \cdot$ ) it follows that this isomorphism lifts to an isomorphism

$$\text{unfold}_{\mu X.A} : \llbracket \mu X.A \rrbracket \cong \mathbb{I}^{\mu X.A} (\llbracket X \vdash A \rrbracket \mathbb{I}^{\mu X.A} (\llbracket \mu X.A \rrbracket)) = \mathbb{I}^{\mu X.A} (\llbracket A[\mu X.A/X] \rrbracket) : \text{fold}_{\mu X.A}$$

in the category  $\mathbf{R}(\mu X.A)$ . Expanding definitions, this means we have an isomorphism

$$\begin{aligned} \text{unfold}_{\mu X.A} : (\llbracket \mu X.A \rrbracket, \mathbf{id}, \triangleleft_{\mu X.A}) &= \llbracket \mu X.A \rrbracket \\ &\cong \mathbb{I}^{\mu X.A} (\llbracket A[\mu X.A/X] \rrbracket) \\ &= (\llbracket A[\mu X.A/X] \rrbracket, \text{fold}_{\mu X.A}, \mathbb{I}^{\mu X.A} \triangleleft_{A[\mu X.A/X]}) : \text{fold}_{\mu X.A} \end{aligned} \tag{13}$$

in the category  $\mathbf{R}(\mu X.A)$ . The notion of morphism in this category (Definition 114), construction of  $\mathbb{I}$  (Definition 128) and property (C5) allow us to conclude that property (A4) is satisfied. Indeed:

$$\begin{aligned} &v \triangleleft_{\mu X.A} \text{fold } V \\ \implies &\text{unfold}_{\mu X.A} \circ v (\mathbb{I}^{\mu X.A} \triangleleft_{A[\mu X.A/X]}) \text{fold } V \\ \implies &\text{unfold}_{\mu X.A} \circ v \triangleleft_{A[\mu X.A/X]} V \end{aligned}$$

and for the other direction of (A4):

$$\begin{aligned} &\text{unfold}_{\mu X.A} \circ v \triangleleft_{A[\mu X.A/X]} V \\ \implies &\text{unfold}_{\mu X.A} \circ v (\mathbb{I}^{\mu X.A} \triangleleft_{A[\mu X.A/X]}) \text{fold } V \\ \implies &v = \text{fold}_{\mu X.A} \circ \text{unfold}_{\mu X.A} \circ v \triangleleft_{\mu X.A} \text{fold } V. \quad \square \end{aligned}$$

### F. Closure Properties of the Logical Relations

Here we establish some important closure properties of the relations  $\overline{\triangleleft}_A$  from Theorem 138.

**Lemma 139.** *Let  $\cdot \vdash M : A$  be a term and let  $F$  be some finite index set. Assume that we are given morphisms  $m_i$  and terms  $M_i$  such that  $m_i \overline{\triangleleft}_A M_i$  for  $i \in F$ . Assume further that for each  $i \in F$ , we are given a reduction path  $\pi_i \in \text{Paths}(M, M_i)$ , such that all paths  $\pi_i$  are distinct. Then*

$$\sum_{i \in F} P(\pi_i) m_i \overline{\triangleleft}_A M.$$

*Proof.* By assumption, for every  $i \in F$ , we know that  $m_i \in \mathcal{S}(\triangleleft_A; M_i)$ . Next, consider the function

$$g \stackrel{\text{def}}{=} \sum_{i \in F} P(\pi_i)(-): \prod_{|F|} \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket).$$

This function is Scott continuous and therefore by Lemma 107, it suffices to show that  $g(\prod_i s_i) \in \mathcal{S}(\triangleleft_A; M)$  for any choice of  $s_i \in \mathcal{S}_0(\triangleleft_A; M_i)$ . Next, for every  $i \in F$ , let

$$s_i = \left( \sum_{\pi \in F_i} P(\pi) v_\pi \right) \in \mathcal{S}_0(\triangleleft_A; M_i)$$

where  $F_i \subseteq \text{TPaths}(M_i)$  is a finite subset and such that  $v_\pi \triangleleft_A V_\pi$ , for each  $\pi \in F_i$ . Then, we have

$$\begin{aligned} g\left(\prod_i s_i\right) &= \sum_{i \in F} P(\pi_i) \left( \sum_{\pi \in F_i} P(\pi) v_\pi \right) \\ &= \sum_{i \in F} \sum_{\pi \in F_i} (P(\pi_i) \cdot P(\pi)) v_\pi \\ &= \sum_{i \in F} \sum_{\pi \in F_i} P(\pi_i \pi) v_\pi \\ &\in \mathcal{S}_0(\triangleleft_A; M), \end{aligned}$$

where  $\pi_i \pi \in \text{Paths}(M, V_\pi)$  is the path constructed by concatenating the path  $\pi_i$  to  $\pi$ . □

**Lemma 140.** *If  $m \overline{\triangleleft}_A M$  and  $n \overline{\triangleleft}_A N$ , then  $p \cdot m + (1 - p) \cdot n \overline{\triangleleft}_A M \text{ or}_p N$ .*

*Proof.* This is just a special case of Lemma 139. □

**Lemma 141.** *For  $i \in \{1, 2\}$ : if  $m \overline{\triangleleft}_{A_i} M$ , then  $\mathcal{J}in_i \circ m \overline{\triangleleft}_{A_1+A_2} \text{in}_i M$ .*

*Proof.* Assume, without loss of generality, that  $i = 1$ . By definition we know that  $m \in \mathcal{S}(\triangleleft_{A_1}; M) = \overline{\mathcal{S}_0(\triangleleft_{A_1}; M)}$ . By Lemma 107, it suffices to show

$$\mathcal{J}in_1 \circ \sum_{\pi \in F} P(\pi) v_\pi \in \mathcal{S}(\triangleleft_{A_1+A_2}; \text{in}_1 M)$$

for any  $\sum_{\pi \in F} P(\pi) v_\pi \in \mathcal{S}_0(\triangleleft_{A_1}; M)$ . Since  $(\mathcal{J}in_1 \circ -)$  is linear, we see

$$\mathcal{J}in_1 \circ \sum_{\pi \in F} P(\pi) v_\pi = \sum_{\pi \in F} P(\pi) (\mathcal{J}in_1 \circ v_\pi) = \sum_{\pi \in F} P(\text{in}_1(\pi)) (\mathcal{J}in_1 \circ v_\pi) \in \mathcal{S}(\triangleleft_{A_1+A_2}; \text{in}_1 M),$$

where  $\text{in}_1(\pi) \in \text{Paths}(\text{in}_1 M, \text{in}_1 V_\pi)$  is the path constructed by reducing  $\text{in}_1 M$  to  $\text{in}_1 V_\pi$ , as specified by  $\pi$ . The membership relation is satisfied because by assumption  $v_\pi \triangleleft_{A_1} V_\pi$  and then by Theorem 138 (A1). □

**Lemma 142.** *Let  $m \overline{\triangleleft}_{A_1+A_2} M$ . Next, assume that for  $k \in \{1, 2\}$  we have terms  $x_k : A_k \vdash N_k : B$  and morphisms  $n_k : \llbracket A_k \rrbracket \rightarrow \llbracket B \rrbracket$ , such that for every  $v_k \triangleleft_{A_k} V_k$ , it is the case that  $n_k \circ v_k \overline{\triangleleft}_B N_k[V_k/x_k]$ . Then*

$$[n_1, n_2] \circ m \overline{\triangleleft}_B \text{case } M \text{ of } \text{in}_1 x_1 \Rightarrow N_1 \mid \text{in}_2 x_2 \Rightarrow N_2.$$

*Proof.* For brevity, let  $C$  be the term  $C \stackrel{\text{def}}{=} (\text{case } M \text{ of } \text{in}_1 x_1 \Rightarrow N_1 \mid \text{in}_2 x_2 \Rightarrow N_2)$ . Next, consider the function

$$([n_1, n_2] \circ -): \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 + A_2 \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket).$$

This function is Scott continuous. By Lemma 107, to complete the proof it suffices to show that  $[n_1, n_2] \circ m' \overline{\triangleleft}_B C$  for any  $m' \in \mathcal{S}_0(\triangleleft_{A_1+A_2}; M)$ . Towards that end, let

$$m' = \sum_{\pi \in F} P(\pi) v_\pi,$$



where  $F$  is finite and where  $v_\pi \triangleleft_{A_1+A_2} V_\pi$ , for each  $\pi \in F$ . Let  $F_1 \subseteq F$  be the set of paths  $\pi$  such that  $V_\pi = \text{in}_1 V'_\pi$  for some  $V'_\pi$  and let  $F_2 = F - F_1$ . Then by Theorem 138 (A1), for each  $\pi \in F_1$ , it follows that  $V_\pi = \text{in}_1 V'_\pi$  and  $v_\pi = \mathcal{J} \text{in}_1 \circ v'_\pi$  and  $v'_\pi \triangleleft_{A_1} V'_\pi$ . Similarly, for each  $\pi \in F_2$ , it follows that  $V_\pi = \text{in}_2 V'_\pi$  and  $v_\pi = \mathcal{J} \text{in}_2 \circ v'_\pi$  and  $v'_\pi \triangleleft_{A_2} V'_\pi$ . Therefore, we get:

$$\begin{aligned} [n_1, n_2] \circ m' &= [n_1, n_2] \circ \left( \left( \sum_{\pi \in F_1} P(\pi)(\mathcal{J} \text{in}_1 \circ v'_\pi) \right) + \left( \sum_{\pi \in F_2} P(\pi)(\mathcal{J} \text{in}_2 \circ v'_\pi) \right) \right) \\ &= \left( \sum_{\pi \in F_1} P(\pi)(n_1 \circ v'_\pi) \right) + \left( \sum_{\pi \in F_2} P(\pi)(n_2 \circ v'_\pi) \right) \end{aligned}$$

In the above sums, by assumption, we know that  $n_1 \circ v'_\pi \overline{\triangleleft}_B N_1[V'_\pi/x_1]$ , for each  $\pi \in F_1$  and similarly  $n_2 \circ v'_\pi \overline{\triangleleft}_B N_2[V'_\pi/x_2]$ , for each  $\pi \in F_2$ . Next, consider the function

$$\left( \left( \sum_{\pi \in F_1} P(\pi)(-) \right) + \left( \sum_{\pi \in F_2} P(\pi)(-) \right) \right) : \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)^{|F_1|} \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)^{|F_2|} \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket).$$

This function is Scott-continuous and by Lemma 107, to complete the proof it suffices to show that

$$\left( \sum_{\pi \in F_1} P(\pi)(n_1^\pi) \right) + \left( \sum_{\pi \in F_2} P(\pi)(n_2^\pi) \right) \overline{\triangleleft}_B C,$$

where  $n_1^\pi \in \mathcal{S}_0(\triangleleft_B; N_1[V'_\pi/x_1])$  for  $\pi \in F_1$  and  $n_2^\pi \in \mathcal{S}_0(\triangleleft_B; N_2[V'_\pi/x_2])$  for  $\pi \in F_2$  are taken to be arbitrary. Towards this end, let

$$\begin{aligned} n_1^\pi &= \sum_{\pi' \in F_1^\pi} P(\pi') v_{\pi'} \in \mathcal{S}_0(\triangleleft_B; N_1[V'_\pi/x_1]) \\ n_2^\pi &= \sum_{\pi' \in F_2^\pi} P(\pi') v_{\pi'} \in \mathcal{S}_0(\triangleleft_B; N_2[V'_\pi/x_2]) \end{aligned}$$

where  $F_k^\pi$  is finite and where  $v_{\pi'} \triangleleft_B V_{\pi'}$ , for every  $\pi' \in F_k^\pi$  and where  $k \in \{1, 2\}$ . Then, we get

$$\begin{aligned} &\left( \sum_{\pi \in F_1} P(\pi)(n_1^\pi) \right) + \left( \sum_{\pi \in F_2} P(\pi)(n_2^\pi) \right) = \\ &= \left( \sum_{\pi \in F_1} \sum_{\pi' \in F_1^\pi} P(\pi) P(\pi') v_{\pi'} \right) + \left( \sum_{\pi \in F_2} \sum_{\pi' \in F_2^\pi} P(\pi) P(\pi') v_{\pi'} \right) \\ &= \left( \sum_{\pi \in F_1} \sum_{\pi' \in F_1^\pi} P(\text{case}_1(\pi, \pi')) v_{\pi'} \right) + \left( \sum_{\pi \in F_2} \sum_{\pi' \in F_2^\pi} P(\text{case}_2(\pi, \pi')) v_{\pi'} \right) \\ &\in \mathcal{S}_0(\triangleleft_B; C) \subseteq \mathcal{S}(\triangleleft_B; C), \end{aligned}$$

where  $\text{case}_1(\pi, \pi') \in \text{Paths}(C, V_{\pi'})$  is the path obtained by reducing  $C$  to  $C_\pi \stackrel{\text{def}}{=} (\text{case in}_1 V'_\pi \text{ of } \text{in}_1 x_1 \Rightarrow N_1 \mid \text{in}_2 x_2 \Rightarrow N_2)$  as specified by  $\pi$ , then performing the beta reduction  $C_\pi \xrightarrow{1} N_1[V'_\pi/x_1]$  and then reducing  $N_1[V'_\pi/x_1]$  to  $V_{\pi'}$  as specified by  $\pi'$ . Similarly for  $\text{case}_2(\pi, \pi')$ . The last sum is now by definition in  $\mathcal{S}_0(\triangleleft_B; C)$ .  $\square$

**Lemma 143.** *If  $m_1 \overline{\triangleleft}_{A_1} M_1$  and  $m_2 \overline{\triangleleft}_{A_2} M_2$  then  $\langle\langle m_1, m_2 \rangle\rangle \overline{\triangleleft}_{A_1 \times A_2} (M_1, M_2)$ .*

*Proof.* The map  $\langle\langle -, - \rangle\rangle : \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 \rrbracket) \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_2 \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 \times A_2 \rrbracket)$  is Scott-continuous in both arguments and therefore by Lemma 107, to complete the proof it suffices to show that  $\langle\langle m'_1, m'_2 \rangle\rangle \overline{\triangleleft}_{A_1 \times A_2} (M_1, M_2)$  for any  $m'_1 \in \mathcal{S}_0(\triangleleft_{A_1}; M_1)$  and  $m'_2 \in \mathcal{S}_0(\triangleleft_{A_2}; M_2)$ .

Now, take  $m'_1 = \sum_{\pi_1 \in F_1} P(\pi_1)v_{\pi_1} \in \mathcal{S}_0(\triangleleft_{A_1}; M_1)$  and  $m'_2 = \sum_{\pi_2 \in F_2} P(\pi_2)v_{\pi_2} \in \mathcal{S}_0(\triangleleft_{A_2}; M_2)$ , where  $F_1$  and  $F_2$  are finite sets, and where  $v_{\pi_1} \triangleleft_{A_1} V_{\pi_1}$  for each  $\pi_1 \in F_1$  and where  $v_{\pi_2} \triangleleft_{A_2} V_{\pi_2}$  for each  $\pi_2 \in F_2$ . We then have:

$$\langle\langle m'_1, m'_2 \rangle\rangle = \langle\langle \sum_{\pi_1 \in F_1} P(\pi_1)v_{\pi_1}, \sum_{\pi_2 \in F_2} P(\pi_2)v_{\pi_2} \rangle\rangle \quad (14)$$

$$= \sum_{\pi_1 \in F_1} \sum_{\pi_2 \in F_2} P(\pi_1)P(\pi_2)\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \quad (15)$$

$$= \sum_{\pi_1 \in F_1} \sum_{\pi_2 \in F_2} P(\text{pair}(\pi_1, \pi_2))\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \quad (16)$$

$$\overline{\triangleleft_{A_1 \times A_2}} (M_1, M_2). \quad (17)$$

Equation 14 holds by definition. Equation 15 is true since the function  $\langle\langle -, - \rangle\rangle$  defined above is linear in each component by Lemma 36 Item 3. In Equation 16  $\text{pair}(\pi_1, \pi_2) \in \text{Paths}((M_1, M_2), (V_{\pi_1}, V_{\pi_2}))$  is the path which first reduces  $(M_1, M_2)$  to  $(V_{\pi_1}, M_2)$  as specified by  $\pi_1$  and then reduces  $(V_{\pi_1}, M_2)$  to  $(V_{\pi_1}, V_{\pi_2})$  as specified by  $\pi_2$  and it is easy to see that Equation 16 holds. Finally 17 holds, because  $v_{\pi_1} \triangleleft_{A_1} V_{\pi_1}$  and  $v_{\pi_2} \triangleleft_{A_2} V_{\pi_2}$  by assumption and then by Theorem 138 (A2) we have that  $\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \triangleleft_{(A_1, A_2)} (V_{\pi_1}, V_{\pi_2})$ .  $\square$

**Lemma 144.** *If  $m \overline{\triangleleft_{A_1 \times A_2}} M$  then  $\mathcal{J}\pi_i \circ m \overline{\triangleleft_{A_i}} \pi_i M$ , for  $i \in \{1, 2\}$ .*

*Proof.* Without loss of generality, we will show the statement for the first projection. In order to avoid notational confusion, we will write  $\text{pr}_1$  for  $\pi_1$  for the projection on the first component in this lemma. We shall use  $\pi$  to range over paths, as in the other lemmas.

Using Lemma 107, to complete the proof it suffices to show that

$$\mathcal{J}\text{pr}_1 \circ m' \overline{\triangleleft_{A_1}} \text{pr}_1 M$$

for any  $m' \in \mathcal{S}_0(\triangleleft_{A_1 \times A_2}; M)$ . Towards this end, let

$$m' = \sum_{\pi \in F} P(\pi)v_{\pi} \in \mathcal{S}_0(\triangleleft_{A_1 \times A_2}; M),$$

where  $F \subseteq \text{TPaths}(M)$  is finite and where  $v_{\pi} \triangleleft_{A_1 \times A_2} V_{\pi}$  for every  $\pi \in F$ . Using Theorem 138 (A2), we see that it must be the case

$$v_{\pi} = \langle\langle v_{\pi}^1, v_{\pi}^2 \rangle\rangle \text{ and } V_{\pi} = (V_{\pi}^1, V_{\pi}^2) \text{ and } v_{\pi}^1 \triangleleft_{A_1} V_{\pi}^1 \text{ and } v_{\pi}^2 \triangleleft_{A_2} V_{\pi}^2.$$

Therefore, we have

$$\begin{aligned} \mathcal{J}\text{pr}_1 \circ m' &= \mathcal{J}\text{pr}_1 \circ \sum_{\pi \in F} P(\pi)v_{\pi} \\ &= \mathcal{J}\text{pr}_1 \circ \sum_{\pi \in F} P(\pi)\langle\langle v_{\pi}^1, v_{\pi}^2 \rangle\rangle \\ &= \sum_{\pi \in F} P(\pi)(\mathcal{J}\text{pr}_1 \circ \langle\langle v_{\pi}^1, v_{\pi}^2 \rangle\rangle) \\ &= \sum_{\pi \in F} P(\pi)v_{\pi}^1 \\ &= \sum_{\pi \in F} P(\text{pr}_1(\pi))v_{\pi}^1 \\ &\overline{\triangleleft_{A_1}} \text{pr}_1 M, \end{aligned}$$

where  $\text{pr}_1(\pi) \in \text{Paths}(\text{pr}_1 M, V_{\pi}^1)$  is the path that reduces  $\text{pr}_1 M$  to  $\text{pr}_1(V_{\pi}^1, V_{\pi}^2)$  as specified by  $\pi$  and then finally performs the reduction  $\text{pr}_1(V_{\pi}^1, V_{\pi}^2) \xrightarrow{1} V_{\pi}^1$ .  $\square$

**Lemma 145.** *If  $m \overline{\triangleleft_{\mu X.A}} M$  then  $\text{unfold} \circ m \overline{\triangleleft_{A[\mu X.A/X]}} \text{unfold } M$ .*

*Proof.* By Lemma 107, to complete the proof it suffices to show that

$$\text{unfold} \circ m' \in \mathcal{S}(\triangleleft_{A[\mu X.A/X]}; \text{unfold } M)$$

for any  $m' \in \mathcal{S}_0(\triangleleft_{\mu X.A}; M)$ . Towards this end, let

$$m' = \sum_{\pi \in F} P(\pi)v_{\pi} \in \mathcal{S}_0(\triangleleft_{\mu X.A}; M)$$

for some finite  $F \subseteq \text{TPaths}(M)$  and where  $v_\pi \triangleleft_{\mu X.A} V_\pi = \text{fold } V'_\pi$  for each  $\pi \in F$ . Then we have

$$\begin{aligned} \text{unfold} \circ m' &= \sum_{\pi \in F} P(\pi)(\text{unfold} \circ v_\pi) \\ &= \sum_{\pi \in F} P(\text{unfold}(\pi))(\text{unfold} \circ v_\pi) \\ &\in \mathcal{S}_0(\triangleleft_{A[\mu X.A/X]}; \text{unfold } M), \end{aligned}$$

where  $\text{unfold}(\pi) \in \text{Paths}(\text{unfold } M, V'_\pi)$  is the path that reduces  $\text{unfold } M$  to  $\text{unfold fold } V'_\pi$  as specified by  $\pi$  and then finally performs the reduction  $\text{unfold fold } V'_\pi \xrightarrow{1} V'_\pi$ . This last sum satisfies the membership relation, because we know that  $v_\pi \triangleleft_{\mu X.A} V_\pi = \text{fold } V'_\pi$  and then by Theorem 138 (A4) we see that  $\text{unfold} \circ v_\pi \triangleleft_{A[\mu X.A/X]} V'_\pi$ , as required.  $\square$

**Lemma 146.** *If  $m \overline{\triangleleft_{A[\mu X.A/X]}} M$  then  $\text{fold} \circ m \overline{\triangleleft_{\mu X.A}} \text{fold } M$ .*

*Proof.* The function

$$(\text{fold} \circ -): \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A[\mu X.A/X] \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket \mu X.A \rrbracket)$$

is Scott-continuous and therefore by Lemma 107, to complete the proof it suffices to show that

$$\text{fold} \circ m' \in \mathcal{S}(\triangleleft_{\mu X.A}; \text{fold } M)$$

for each  $m' \in \mathcal{S}_0(\triangleleft_{A[\mu X.A/X]}; M)$ . Towards this end, assume that

$$m' = \sum_{\pi \in F} P(\pi)v_\pi \in \mathcal{S}_0(\triangleleft_{A[\mu X.A/X]}; M),$$

where  $F \subseteq \text{TPaths}(M)$  is finite and for each  $\pi \in F$  we have  $v_\pi \triangleleft_{A[\mu X.A/X]} V_\pi$ . Therefore, by Theorem 138 (A4) we conclude that  $\text{fold} \circ v_\pi \triangleleft_{\mu X.A} \text{fold } V_\pi$ , for each  $\pi \in F$ . Now we finish the proof with the following derivation:

$$\begin{aligned} \text{fold} \circ m' &= \text{fold} \circ \sum_{\pi \in F} P(\pi)v_\pi \\ &= \sum_{\pi \in F} P(\pi)(\text{fold} \circ v_\pi) \\ &= \sum_{\pi \in F} P(\text{fold}(\pi))(\text{fold} \circ v_\pi) \\ &\in \mathcal{S}_0(\triangleleft_{\mu X.A}; \text{fold } M) \subseteq \mathcal{S}(\triangleleft_{\mu X.A}; \text{fold } M), \end{aligned}$$

where  $\text{fold}(\pi) \in \text{Paths}(\text{fold } M, \text{fold } V_\pi)$  is the path that reduces  $\text{fold } M$  to  $\text{fold } V_\pi$  as specified by  $\pi$ .  $\square$

**Lemma 147.** *If  $m \overline{\triangleleft_{A \rightarrow B}} M$  and  $n \overline{\triangleleft_A} N$ , then  $m[n] \overline{\triangleleft_B} MN$ .*

*Proof.* Consider the function  $g: \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rightarrow B \rrbracket) \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)$  defined by  $g(x, y) = x[y]$  (see Notation 120). This function is Scott continuous and linear in both arguments. By Lemma 107, to complete the proof it suffices to show that  $m'[n'] \overline{\triangleleft_B} MN$  for any  $m' \in \mathcal{S}_0(\triangleleft_{A \rightarrow B}; M)$  and  $n' \in \mathcal{S}_0(\triangleleft_A; N)$ . Towards that end, let

$$\begin{aligned} m' &= \sum_{\pi \in F} P(\pi)v_\pi \in \mathcal{S}_0(\triangleleft_{A \rightarrow B}; M) \\ n' &= \sum_{\pi' \in F'} P(\pi')v_{\pi'} \in \mathcal{S}_0(\triangleleft_A; N) \end{aligned}$$

with  $v_\pi \triangleleft_{A \rightarrow B} V_\pi$  and  $v_{\pi'} \triangleleft_A V_{\pi'}$ . Then by Theorem 138 (A3) we have that  $v_\pi[v_{\pi'}] \overline{\triangleleft_B} V_\pi V_{\pi'}$  and

$$\begin{aligned} m'[n'] &= \sum_{\pi \in F} \sum_{\pi' \in F'} (P(\pi) \cdot P(\pi')) v_\pi[v_{\pi'}] \\ &= \sum_{(\pi, \pi') \in F \times F'} P(\text{app}(\pi, \pi')) v_\pi[v_{\pi'}] \\ &\overline{\triangleleft_B} MN \end{aligned} \tag{Lemma 139}$$

where  $\text{app}(\pi, \pi') \in \text{Paths}(MN, V_\pi V_{\pi'})$  is the path where we first reduce  $MN$  to  $V_\pi N$  in the same way as in  $\pi$  and then we reduce  $V_\pi N$  to  $V_\pi V_{\pi'}$  in the same way as in  $\pi'$ . Note: in the above sum  $V_\pi V_{\pi'}$  is not a value, so Lemma 139 is crucial.  $\square$

### G. Fundamental Lemma and Strong Adequacy

We may now prove the Fundamental Lemma which then easily implies our adequacy result.

**Lemma 148** (Fundamental). *Let  $x_1 : A_1, \dots, x_n : A_n \vdash M : B$  be a term. Assume further we are given a collection of morphisms  $v_i$  and values  $V_i$ , such that  $v_i \triangleleft_{A_i} V_i$  for  $i \in \{1, \dots, n\}$ . Then:*

$$\llbracket M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle \overline{\triangleleft}_B M[\vec{V}/\vec{x}].$$

*Proof.* By induction on the derivation of the term  $M$ .

For the case of lambda abstractions, we reason as follows. Let us assume that the term of the induction hypothesis is

$$x_1 : A_1, \dots, x_n : A_n, y : A \vdash M : B.$$

Let us write  $l \stackrel{\text{def}}{=} \llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle$  and  $R \stackrel{\text{def}}{=} \lambda y. M[\vec{V}/\vec{x}]$ . Observe that  $l \in \mathbf{TD}$  and therefore by Theorem 138 (C5), we may equivalently show that

$$l \triangleleft_{A \rightarrow B} R.$$

By Theorem 138 (A3), this is in turn equivalent to showing that

$$l \leq \llbracket R \rrbracket \text{ and } \forall (w \triangleleft_A W). l[w] \overline{\triangleleft}_B RW.$$

The inequality is satisfied, because

$$\begin{aligned} l &= \llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle \\ &\leq \llbracket \lambda y. M \rrbracket \circ \langle \langle \llbracket \vec{V} \rrbracket \rangle \rangle && \text{(Theorem 138 (C1))} \\ &= \llbracket R \rrbracket. && \text{(Lemma 63)} \end{aligned}$$

For the other requirement, assuming that  $w \triangleleft_A W$ , we reason as follows

$$\begin{aligned} l[w] &= (\llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle)[w] && \text{(Definition)} \\ &= \epsilon \circ (\llbracket \lambda y. M \rrbracket \times \mathbf{id}) \circ \langle \langle \vec{v}, w \rangle \rangle \\ &= \epsilon \circ (\mathcal{J}\lambda(\llbracket M \rrbracket) \times \mathbf{id}) \circ \langle \langle \vec{v}, w \rangle \rangle && \text{(Definition)} \\ &= \lambda^{-1}(\lambda(\llbracket M \rrbracket)) \circ \langle \langle \vec{v}, w \rangle \rangle && \text{(Property of adjunction (5))} \\ &= \llbracket M \rrbracket \circ \langle \langle \vec{v}, w \rangle \rangle \\ &\overline{\triangleleft}_B M[\vec{V}/\vec{x}, W/y]. && \text{(Induction Hypothesis)} \end{aligned}$$

Finally, observe that  $RW = (\lambda y. M[\vec{V}/\vec{x}])W \xrightarrow{1} M[\vec{V}/\vec{x}, W/y]$ , i.e.  $RW$  beta-reduces to  $M[\vec{V}/\vec{x}, W/y]$ . Therefore by Lemma 139 it follows that

$$l[w] \overline{\triangleleft}_B RW,$$

as required.

The case for variables follows immediately by expanding definitions and Theorem 138 (C5).

All other cases follow by straightforward induction using closure Lemmas 140 – 147. □

Adequacy now follows as a corollary of this lemma.

**Theorem 149** (Strong Adequacy). *For any closed term  $\cdot \vdash M : A$ , we have*

$$\llbracket M \rrbracket = \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

*Proof.* Let

$$u \stackrel{\text{def}}{=} \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

From Corollary 98, we know that  $\llbracket M \rrbracket \geq u$ . To finish the proof, we have to show the converse inequality. Next, observe that  $\mathcal{S}_0(\triangleleft_A; M) \subseteq \downarrow u$ , which follows from Theorem 138 (C1). To see this, we reason as follows. Taking an arbitrary element of  $\mathcal{S}_0(\triangleleft_A; M)$  as in Theorem 138 (B):

$$\begin{aligned}
\sum_{\pi \in F} P(\pi)v_\pi &\leq \sum_{\pi \in F} P(\pi)\llbracket V_\pi \rrbracket && \text{(Theorem 138 (C1))} \\
&= \sum_{V \in \cup\{V_\pi \mid \pi \in F\}} \left( \sum_{\substack{\pi \in F \\ V_\pi = V}} P(\pi) \right) \llbracket V \rrbracket \\
&\leq \sum_{V \in \cup\{V_\pi \mid \pi \in F\}} \left( \sum_{\pi \in \text{Paths}(M, V)} P(\pi) \right) \llbracket V \rrbracket \\
&= \sum_{V \in \cup\{V_\pi \mid \pi \in F\}} P(M \rightarrow_* V) \llbracket V \rrbracket \\
&\leq \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.
\end{aligned}$$

The set  $\downarrow u$  is Scott-closed and therefore  $\mathcal{S}(\triangleleft_A; M) \subseteq \downarrow u$ . By Lemma 148, we know that  $\llbracket M \rrbracket \overline{\triangleleft}_A M$ . By definition of  $\overline{\triangleleft}_A$  it follows  $\llbracket M \rrbracket \in \mathcal{S}(\triangleleft_A; M)$  and therefore  $\llbracket M \rrbracket \leq u$ , thus finishing the proof.  $\square$