Rewriting Families of String Diagrams

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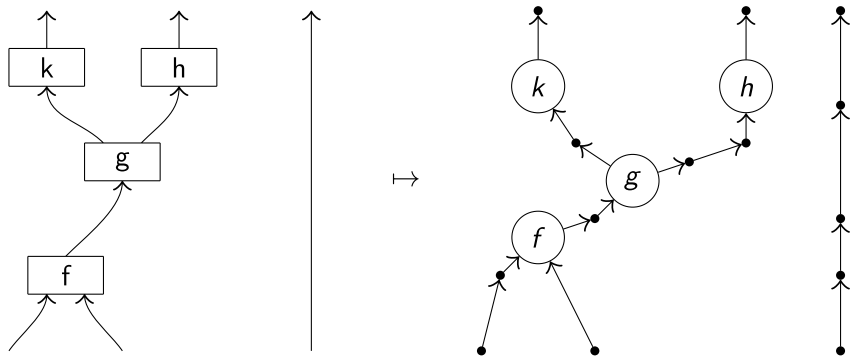
Introduction

- String diagrams have found applications in many areas (quantum computing, petri nets, etc.).
- Equational reasoning with string diagrams may be automated (Quantomatic).
- Reasoning for *families* of string diagrams is sometimes necessary (verifying quantum protocols/algorithms).
- In this talk we will describe a framework which allows us to rewrite context-free families of string diagrams.

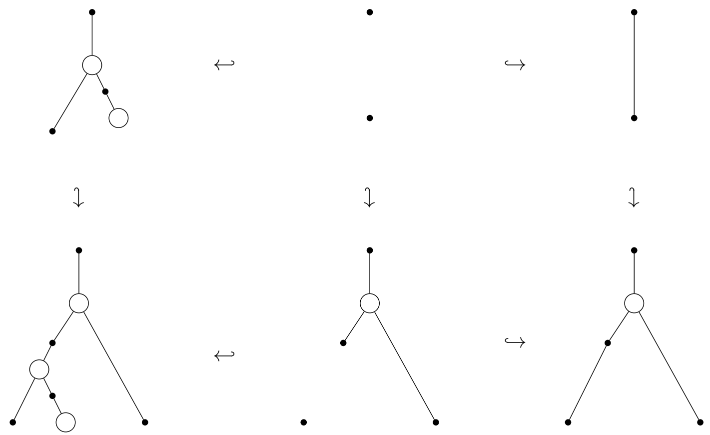
String Diagrams and String Graphs



- Discrete representation exists in the form of *String Graphs*
- String graphs are typed (directed) graphs, such that:
  - Every vertex is either a *node-vertex* or a *wire-vertex*
  - No edges between node-vertices
  - In-degree of every wire-vertex is at most one
  - Out-degree of every wire-vertex is at most one

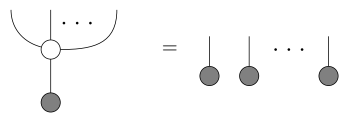
Reasoning with String Graphs

We use double-pushout (DPO) rewriting on string graphs to represent string diagram rewriting:



Families of string diagrams

Example

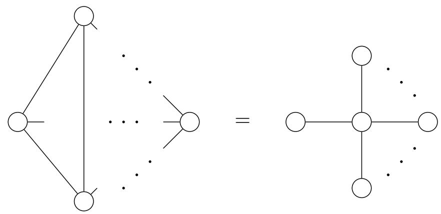


Motivation

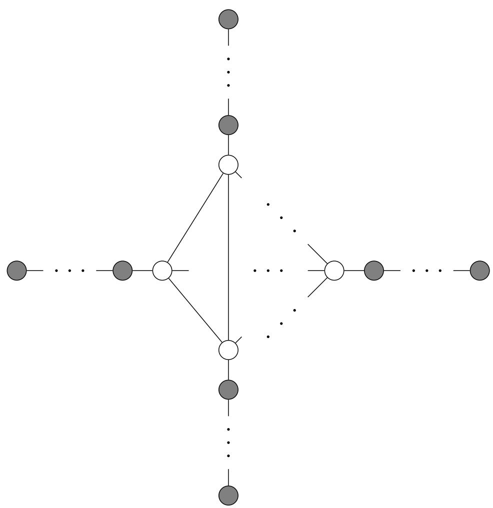
- Given an equational schema between two families of string diagrams, how can we apply it to a target family of string diagrams and obtain a new equational schema?

Example

Equational schema between complete graphs on n vertices and star graphs on n vertices:

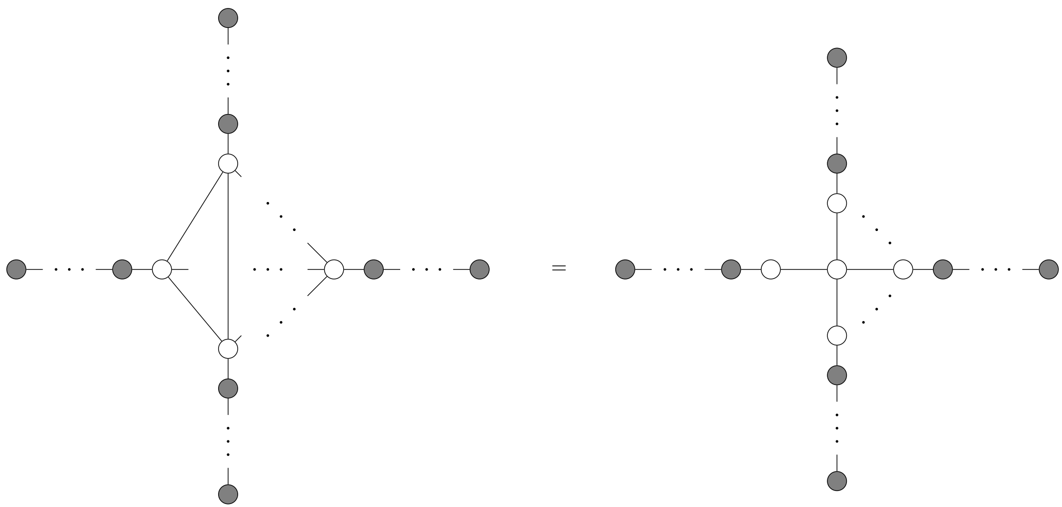


Then, we can apply this schema to the following family of graphs:



Motivation

and we obtain a new equational schema:



The main ideas are:

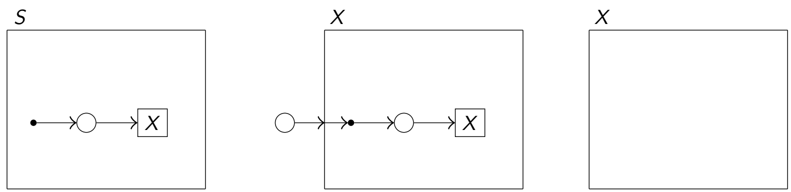
- Context-free graph grammars represent families of graphs
- Grammar DPO rewrite rules represent equational schemas
- Grammar DPO rewriting represents equational reasoning on families of graphs
- Grammar DPO rewriting is admissible (or correct) w.r.t. concrete instantiations

Context-free graph grammars

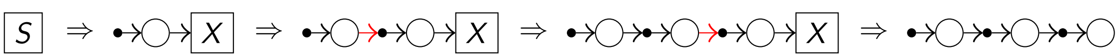
- We will be using context-free graph grammars to represent families of (string) graphs
- Large body of literature available

Example

The following grammar generates the set of all chains of node vertices with an input and no outputs:



A derivation in the above grammar of the string graph with three node vertices:



where we color the newly established edges in red.

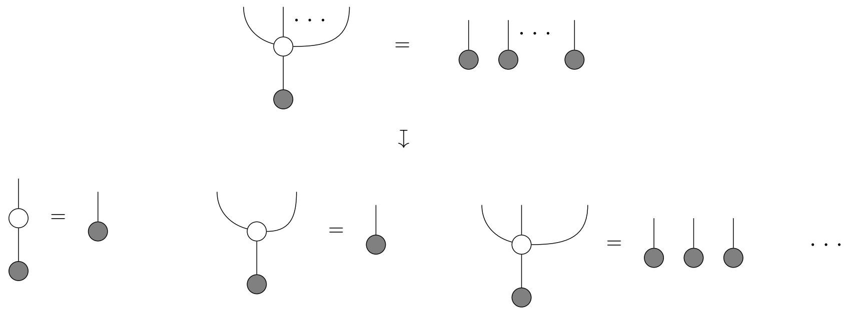
- A context-free grammar is a graph-like structure – essentially it is a partition of graphs equipped with connection instructions

Adhesivity of edNCE grammars

- The category of context-free grammars **GGram** is a partially adhesive category
- Suitable for performing DPO rewriting
- DPO rewriting along with gluing conditions in **GGram** are straightforward generalisations of the standard DPO method
- Languages induced by context-free grammars are defined set-theoretically, not algebraically
- Restrictions on rewrite rules and matchings necessary if we wish rewriting in **GGram** to make sense w.r.t language generation

Quantification over equalities

- an equational schema between two families of string diagrams establishes infinitely many equalities:



- How do we model this using edNCE grammars?
- Idea: DPO rewrite rule in **GGram**, where productions are in 1-1 correspondance

Grammar rewrite pattern

Definition (Grammar rewrite pattern)

A *Grammar rewrite pattern* is a triple of grammars B_L, B_I and B_R, such that there is a bijection between their productions which also preserves non-terminals and their labels.

Definition (Pattern instantiation)

Given a grammar rewrite pattern (B_L, B_I, B_R), a pattern instantiation is given by a triple of concrete derivations:

$$S \Longrightarrow^{B_L}_{v_1,p_1} H_1 \Longrightarrow^{B_L}_{v_2,p_2} H_2 \Longrightarrow^{B_L}_{v3,p3} \cdots \Longrightarrow^{B_L}_{v_n,p_n} H_n$$

and

$$S \Longrightarrow^{B_I}_{v_1,p_1} H_1' \Longrightarrow^{B_I}_{v_2,p_2} H_2' \Longrightarrow^{B_I}_{v3,p3} \cdots \Longrightarrow^{B_I}_{v_n,p_n} H_n'$$

and

$$S \Longrightarrow^{B_R}_{v_1,p_1} H_1'' \Longrightarrow^{B_R}_{v_2,p_2} H_2'' \Longrightarrow^{B_R}_{v3,p3} \cdots \Longrightarrow^{B_R}_{v_n,p_n} H_n''$$

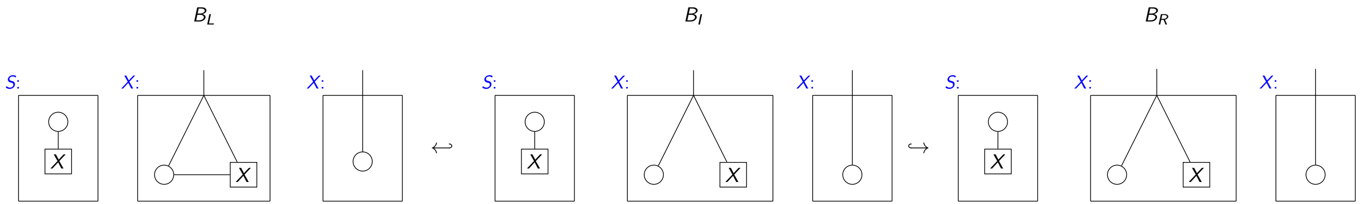
- That is, we always expand the same non-terminals in the three sentential forms in parallel

Theorem

Every pattern instantiation is a DPO rewrite rule on graphs.

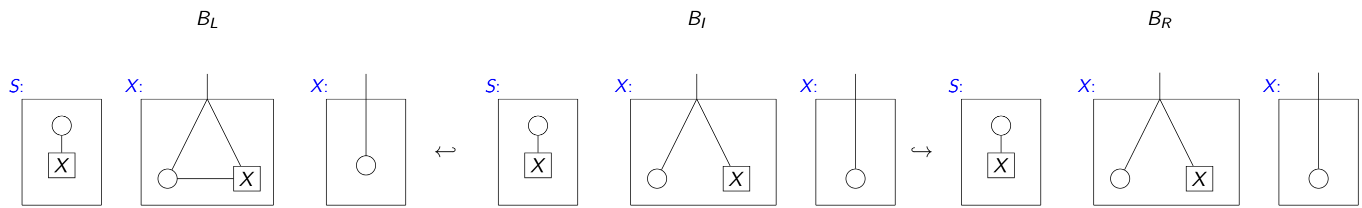
Grammar rewrite pattern

Example



Grammar rewrite pattern

Example



- Instantiation :

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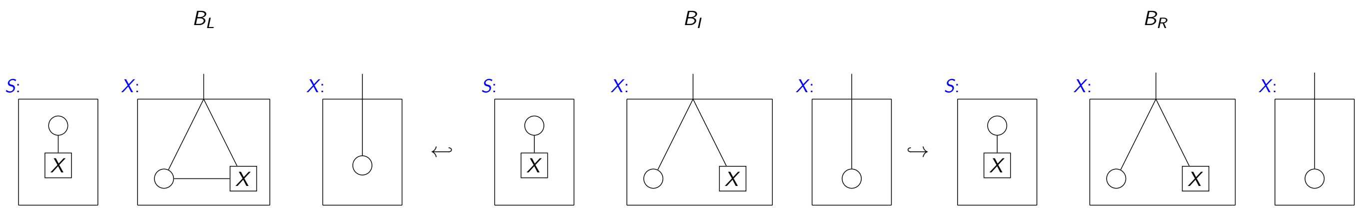
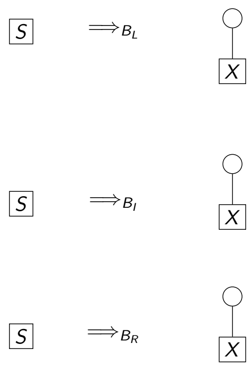
Grammar rewrite pattern

Example

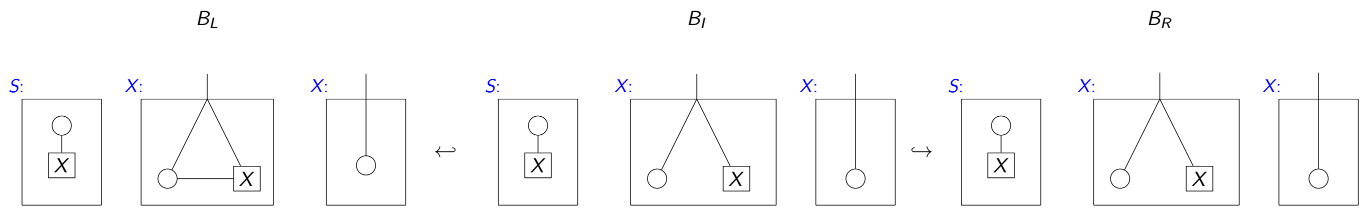


- Instantiation :

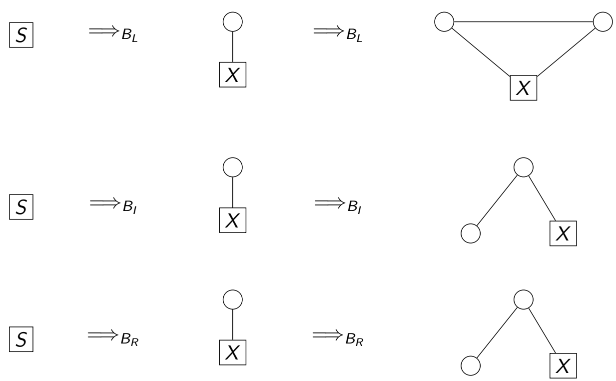


Grammar rewrite pattern

Example

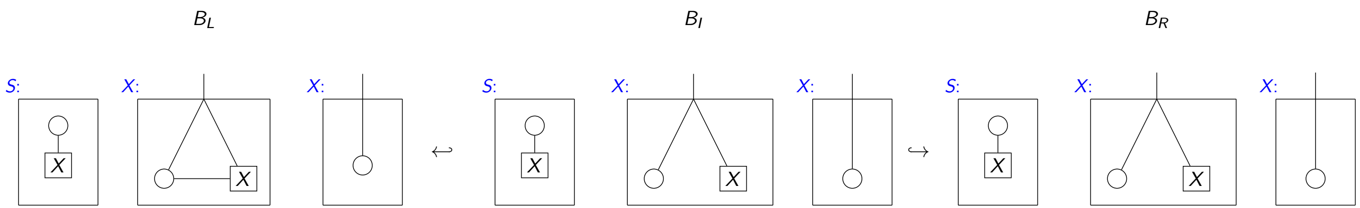


- Instantiation :

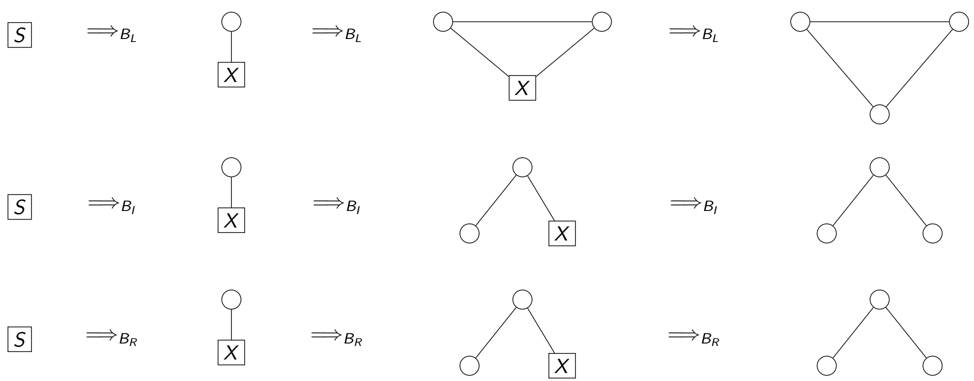


Grammar rewrite pattern

Example

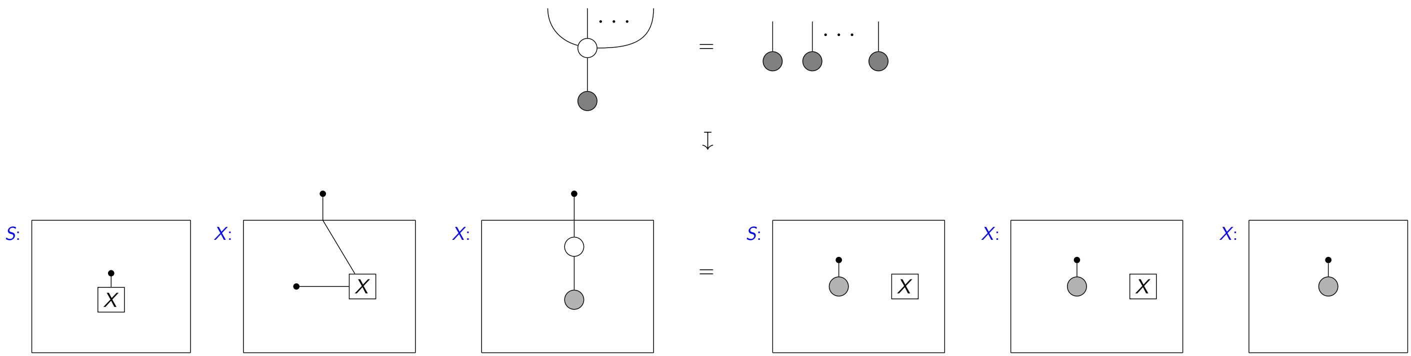


- Instantiation :



Obtaining new equalities

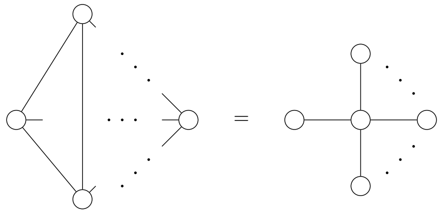
- We can encode infinitely many equalities between string diagrams by using grammar rewrite patterns



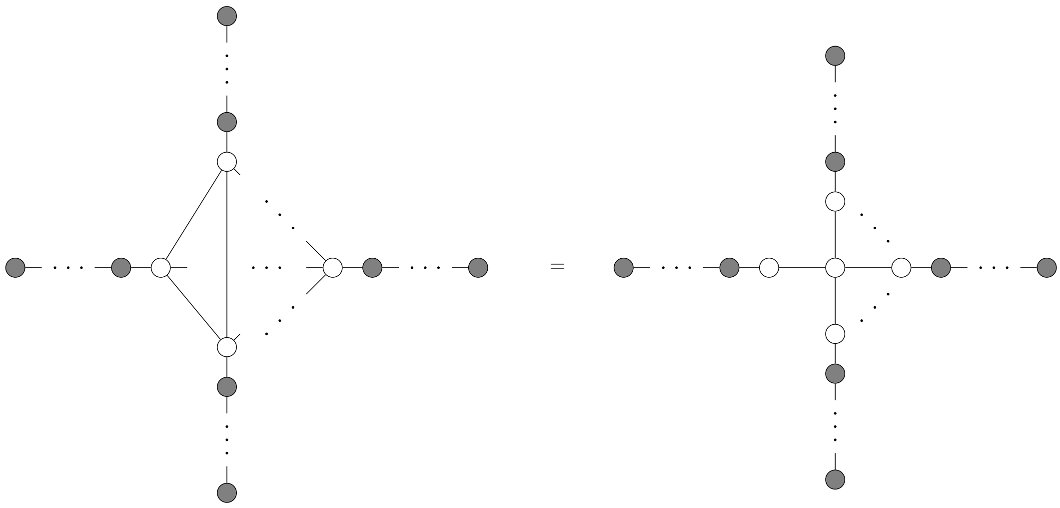
- Next, we show how to rewrite a family of diagrams using an equational schema in an admissible way

Example

Given an equational schema:

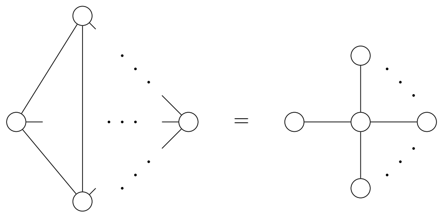


how do we apply it to a target family of string diagrams (left) and get the resulting family (right):

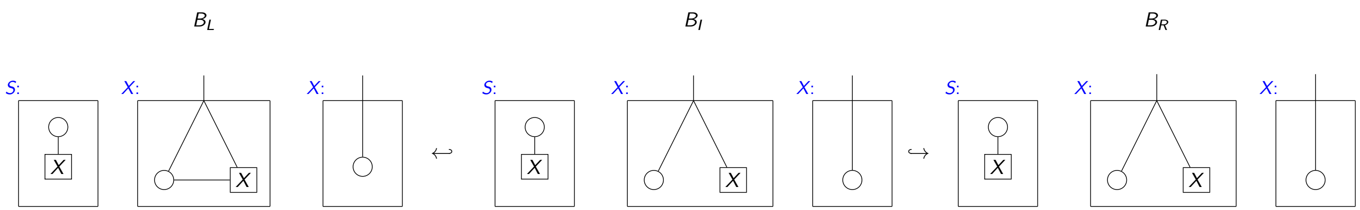


Step one

Encode equational schema as a grammar rewrite pattern.
This:



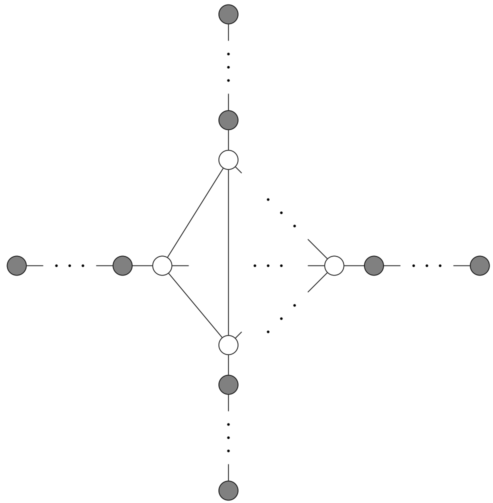
becomes this:



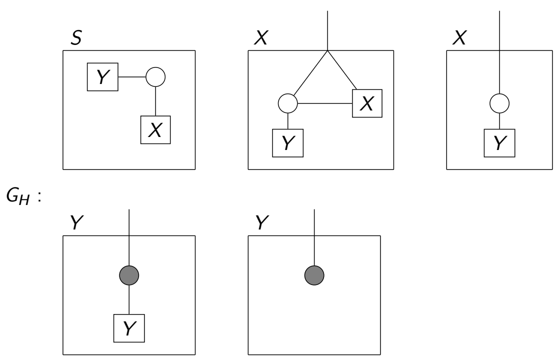
Step two

Encode the target family of string diagrams using a grammar

This:



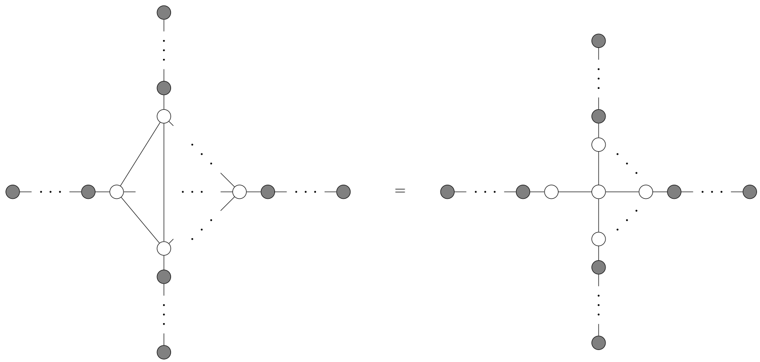
becomes this:



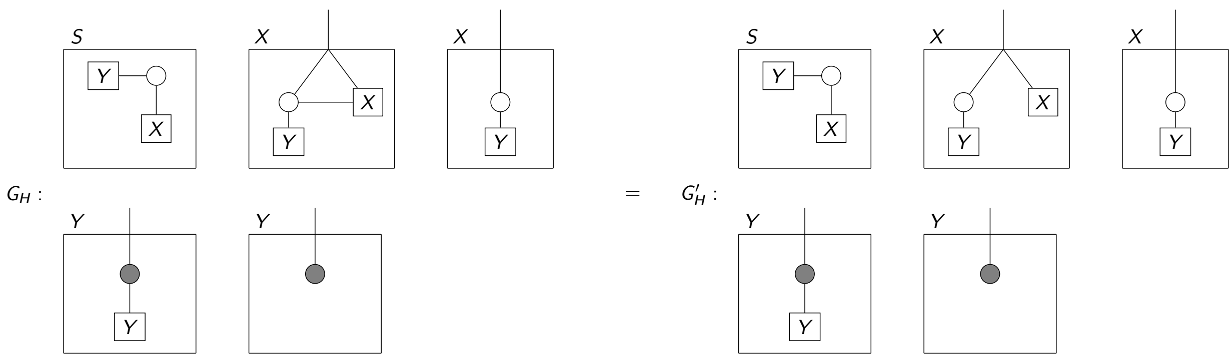
Step three

- Match the grammar rewrite rule into the target grammar and perform DPO rewrite (in **GGram**)
- Note, both the rewrite rules and the matchings are more restricted than what is required by adhesivity in order to ensure admissibility

This:



is then given by:



Admissibility

- Grammar rewriting as defined is admissible in the sense that it respects the concrete semantics of the grammars
- More formally:
- If a grammar G rewrites into a grammar G' via a grammar rewrite rule B, then:
  - Every concrete instantiation of B is a standard DPO rewrite rule on graphs
  - The language of B, denoted $L(B)$ is the set of all such DPO rewrite rules
  - The pair (G, G') forms a grammar pattern
  - For any concrete instantion H of G, a parallel concrete derivation H' exists for G'.
  - Then, the graph H' can be obtained from the graph H by applying some number of DPO rewrite rules on graphs from $L(B)$ in any order

Conclusion and Future Work

- Basis for formalized equational reasoning for context-free families of string diagrams.
  - Framework can handle equational schemas and it can apply them to equationally reason about families of string diagrams
- Implementation in software (e.g. Quantomatic proof assistant)
- Once implemented, software tools can be used for automated reasoning for quantum computation, petri nets, etc.
- Consider program optimization for circuit description languages
  - I am currently working on a denotational model for a string diagram programming language.

Thank you for your attention!